# Green's theorem and isolation in planar graphs

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## ABSTRACT

We show a simple application of Green's theorem from multivariable calculus to the isolation problem in planar graphs. In particular, we give a log-space construction of a skew-symmetric, polynomially-bounded edge weight function for directed planar graphs, such that the weight of any simple cycle in the graph is non-zero with respect to this weight function. As a direct consequence of the above weight function, we are able to isolate a directed path between two fixed vertices, in a directed planar graph. We also show that given a bipartite planar graph, we can obtain an edge weight function (using the above function) in log-space, which isolates a perfect matching in the given graph. Earlier this was known to be true only for grid graphs – which is a proper subclass of planar graphs.

We also look at the problem of obtaining a straight line embedding of a planar graph in log-space. Although we do not quite achieve this goal, we give a piecewise straight line embedding of the given planar graph in log-space.

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## 1. Introduction

We show a simple application of a celebrated theorem due to 19th century British mathematician George Green to the isolation problems in planar graphs. Green's theorem, stated below, relates certain line integrals over a closed curve on the plane to a related double integral over the region enclosed by this curve.

**Theorem 1** (*Green's theorem*). *Let $C$ be a closed, piecewise smooth, simple curve on the plane which is oriented counterclockwise. Let $R_C$ be the region bounded by $C$. Let $P$ and $Q$ be functions of $(x, y)$ defined on a region containing $R_C$ and having continuous partial derivatives in the region. Then*

$$\oint_C (P\,dx + Q\,dy) = \iint_{R_C} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

This fundamental theorem and its generalizations (such as Stokes' theorem) have deeply influenced the development of several areas of physics and mathematics. Strikingly, Green's theorem also has a very immediate and elegant practical application in calculating the area of an arbitrary two-dimensional shape. The device known as planimeter, used to calculate the area of an arbitrary shape (such as a region in a map) is based on the following instantiation of Green's theorem, which we also use in this paper. If we substitute $Q(x, y) = x$ and $P(x, y) = 0$ in Green's theorem we get the following theorem.

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**Theorem 2** (*Area by line integrals*). *Let $C$ be a closed, piecewise smooth, simple curve on the plane which is oriented counterclockwise. Let $R_C$ be the region bounded by $C$. Then,*

$$\text{Area}(R_C) = \oint_C x\, dy.$$

Refer to any standard textbook on calculus (such as [1]) to know more about Green's and other related theorems.

Distinguishing a solution out of a set of solutions is a basic algorithmic problem with many applications. The well-known *isolating lemma* (also known as the isolation lemma) due to Mulmuley, Vazirani, and Vazirani provides a general randomized solution to this problem. Let $\mathcal{F}$ be a non-empty set system on a universe $U = \{1, \ldots, n\}$. Then the *isolating lemma* says, for a random weight function on $U$ (bounded by $n^{O(1)}$), with high probability there is a *unique* set in $\mathcal{F}$ of minimum weight [2]. This lemma, originally used to give an elegant RNC algorithm for constructing a maximum matching (by isolating a minimum weight perfect matching) in general graphs, has found many applications, mostly in discovering new randomized and non-uniform upper bounds, via isolating minimum weight solutions [2–5]. Clearly, derandomizing the isolating lemma in sufficient generality will improve these upper bounds to their deterministic counterparts and hence will be a major result. Unfortunately, it was shown by Arvind and Mukhopadhyay that such a derandomization will imply certain circuit lower bounds and hence is a difficult task [6]. However, such negative results do not rule out the possibility of bypassing the isolating lemma altogether by directly prescribing efficient deterministic weight functions for specific situations so that the minimum weight solutions become unique. In fact, recently simple log-space computable weight functions were prescribed for directed reachability and bipartite perfect matching problems over *grid graphs* to yield new deterministic upper bounds [7,8]. Grid graphs are a restricted class of planar graphs where the graph completely lies on the two-dimensional grid. It was not clear how to extend these weight functions to planar graphs. In this paper we settle that question.

### 1.1. Our results

Given a directed graph $G$ with a planar embedding, we prescribe a skew-symmetric, log-space computable, polynomially bounded weight function $w$ with the property that, with respect to $w$, the weight of any simple cycle in $G$ is non-zero. We then use arguments identical to that in [7] to show that such weight functions isolate directed paths – that is, with respect to such weight functions, between any pair of nodes if there is a path, then there is a unique minimum-weight path. We also show that such weight functions isolate a matching in bipartite graphs (by appropriately directing edges). Our weight function is based on the line integral on the right hand side of Theorem 2.

The weighting scheme that we prescribe works for any "nice" embedding of the graph on the plane. For ease of presentation we will assume that the graph is presented as a straight line embedding, which means each vertex $v$ is given as a point, $(x_v, y_v)$ on the coordinate axes, and an edge $(u, v)$ is a line between points $(x_u, y_u)$ and $(x_v, y_v)$ so that no such lines intersect other than at the vertices. Moreover, we will assume that the coordinates are integer points with values bounded by poly($n$). Existence of such embeddings were known earlier [9–11].

Typically for algorithmic purposes planar graphs are presented in terms of a *combinatorial embedding*. Time efficient algorithms are known that can compute a straight line embedding of a planar graph [10,11] from a combinatorial embedding. Unfortunately, these algorithms require linear space and at present we do not know how to get a space efficient implementation of them. In Section 4 we give a log-space algorithm that gives a *piecewise straight line embedding* of the given planar graph from a combinatorial embedding. This is the first log-space construction known to us, of a piecewise straight line embedding of a given planar graph and might be of independent interest. It will be very clear how the weight function for a straight line embedding can be extended to a piecewise straight line embedding also.

We do not get any new upper bounds for directed planar reachability or planar bipartite matching problems (other than simplified proofs of existing results) since it is known that these problems over planar graphs reduce to their counterparts in grid graphs [12,8] and hence the weight functions known for grid graphs suffice to derive upper bounds for planar versions of these problems. However, we feel that the application of Green's theorem to the isolation problem gives it a new dimension and might yield potential strategies to solve the more general cases.

## 2. The weight function

Let $G = (V, E)$ be a graph with a straight line embedding. Let $e = (u, v)$ be a directed edge directed from $u$ to $v$ where $u$ is identified with the point $(x_u, y_u)$ and $v$ is identified with $(x_v, y_v)$. For such a directed edge, define a weight function $w$ as follows (if $e$ is piecewise straight, we calculate the integral over each piece and sum them up):

$$w(e) = 2 \times \oint_e x\, dy = (y_v - y_u)(x_v + x_u).$$

In order to calculate the second equality, we can use the parametric equation of the line segment which is given by $x(t) = (x_v - x_u)t + x_u$ and $y(t) = (y_v - y_u)t + y_u$ where $t \in [0, 1]$. Notice that if the coordinates of the vertices are polynomially

bounded, this weight function is also polynomially bounded. For any cycle $C$ in $G$, the weight of $C$, $w(C)$, is defined as the sum of the weights of the edges in $C$.

An important property of this weight function is that it is *skew-symmetric*, that is, $w(u, v) = -w(v, u)$. We use this skew-symmetry property in our proofs. We will first show the following lemma which is crucial in proving that this weight function has the required isolation property.

**Lemma 3.** *Let $G$ be a directed planar graph and let $C$ be any directed simple cycle in $G$. Let $R_C$ be the region enclosed by $C$. Then $|w(C)| = 2 \times \mathrm{Area}(R_c)$. In particular, $w(C)$ is non-zero.*

**Proof.** Let $C = (e_1, e_2, \ldots, e_l)$ be a directed cycle oriented counterclockwise. Then we have

$$w(C) = \sum_i w(e_i)$$

$$= 2 \times \sum_i \oint_{e_i} x\,dy$$

$$= 2 \times \oint_C x\,dy$$

$$= 2 \times \mathrm{Area}(R_C).$$

The last equality follows from Theorem 2. If $C$ is oriented clockwise, we get that $w(C) = -2 \times \mathrm{Area}(R_C)$. Hence the lemma.  □

## 3. Isolating paths and matchings in planar graphs

**Theorem 4.** *Let $G$ be a planar directed graph with a straight line embedding. Then with respect to the weight function $w$, for every pair of nodes $u$ and $v$, if there is a directed path from $u$ to $v$, then there is a unique path from $u$ to $v$ of minimum weight.*

**Proof.** Suppose there are $u, v$ so that there are two $u$ to $v$ paths $P_1$ and $P_2$ of minimum weight. We will assume that the paths do not intersect on vertices other than the end points (otherwise we can find two vertices $u'$ and $v'$ along these paths that satisfy this property using a standard cut-and-paste argument and use these vertices instead). We have $w(P_1) = w(P_2)$. Now consider the graph $G'$ which is the same as $G$ except that the path $P_2$ is reversed so that the set of edges $(P_1, P_2^r)$ becomes a simple cycle in $G'$ ($P_2^r$ denotes the reversal of the path $P_2$). Let $C$ denote this cycle. Then $w(C) = w(P_1) + w(P_2^r) = w(P_1) - w(P_2) = 0$. The second equality holds because of the skew-symmetry of the weight function. This contradicts Lemma 3.  □

Now we will consider isolation of matchings in bipartite planar graphs. Since for matching we have undirected graphs, we need to give directions to the edges in order to assign weights. Let $G$ be a bipartite graph. First we compute the bipartition. This can be achieved in log-space by Reingold's reachability algorithm (say using a universal exploration sequence) for undirected graphs [13]. Thus given a vertex $u$, we can decide in log-space whether $u \in L$ or $u \in R$, where $(L, R)$ is a bipartition of $G$. For any undirected edge $\{u, v\}$ so that $u \in L$ and $v \in R$, we first assign direction from $u$ to $v$. Thus in the corresponding directed graph, denoted by $\vec{G}$, all the edges go from $L$ to $R$. Therefore the weight $w(e)$ of an undirected edge $e = \{u, v\}$ with $u$ at $(x_u, y_u)$ and $v$ at $(x_v, y_v)$ is $(y_v - y_u)(x_v + x_u)$, when directed in this way.

**Theorem 5.** *Let $G$ be a planar undirected bipartite graph. Then with respect to the weight function $w$, if there is a perfect matching in $G$, the minimum weight perfect matching in $G$ is unique.*

**Proof.** Suppose the theorem is not true and let $M_1$ and $M_2$ be two minimum-weight matchings so that $w(M_1) = w(M_2)$. Consider $M_1 \oplus M_2$, the symmetric difference of $M_1$ and $M_1$. This is non-empty and is a collection of simple alternating (between $M_1$ and $M_2$) cycles. Let $C$ be one of the cycles. Let $C_1 = C \cap M_1$ and $C_2 = C \cap M_2$. Then we claim that $w(C_1) = w(C_2)$. Suppose $w(C_1) < w(C_2)$ then $(M_2 \setminus C_2) \cup C_1$ will be a matching of weight smaller than that of $M_2$. Let $\vec{C_1}$ and $\vec{C_2}$ be the corresponding set of directed edges. Now consider a directed planar graph $\vec{G}'$ which is the same as $\vec{G}$ except that the directions of all the edges in $C_2$ is reversed. Thus the edges of $\vec{C_1}$, $\vec{C_2}^r$ form a directed cycle $\vec{C}$ in $\vec{G}'$. But $w(\vec{C}) = w(\vec{C_1}) + w(\vec{C_2}^r) = w(\vec{C_1}) - w(\vec{C_2}) = 0$. This contradicts Lemma 3.  □

### 3.1. A sufficient condition for isolating bipartite matching

Note that the above isolation theorems follow, using simple arguments, from a weight function $w$ for directed graphs with the property that weight of any directed cycle is non-zero with respect to $w$. We state a general result that captures the essentials of the above argument for bipartite matching. A similar theorem holds for isolating directed paths also.

For the purpose of this section, we shall think of an edge weight function as an $n \times n$ matrix. More formally, given a directed graph $G$ on $n$ vertices, $w(G)$ is an $n \times n$ matrix, $(i, j)$-th entry is the weight of the edge $(i, j)$ in $G$.

**Definition 1.** Given an undirected graph $H$, let $\mathcal{O}_H$ be the set of all directed graphs $H'$, such that the underlying graph of $H'$ is $H$. For a class of undirected graphs $\mathcal{H}$, define $\mathcal{O}_\mathcal{H} = \bigcup_{H \in \mathcal{H}} \mathcal{O}_H$.

**Theorem 6.** *Let $\mathcal{G}$ be a class of directed graphs and let $w$ be an edge weight function defined for every $G \in \mathcal{G}$ such that for every graph $G \in \mathcal{G}$, $w(G)$ is skew-symmetric and for any cycle $C$ in $G$, $w(C) \neq 0$. Let $\mathcal{H}$ be a class of undirected bipartite graphs, such that $\mathcal{O}_\mathcal{H} \subseteq \mathcal{G}$. Then we can construct a weight function $w'$ in log-space, such that for every graph $H \in \mathcal{H}$, the minimum weight perfect matching in $H$ with respect to $w'$ is unique.*

**Proof.** Given $H$, use Reingold's undirected reachability algorithm [13], to construct a bipartition of $H$, say $L$ and $R$. Now orient the edges of $H$ as follows to get the graph $H'$: for every edge $e = \{u, v\}$ in $H$, where $u \in L$ and $v \in R$, replace $e$ with the directed edge $e' = (u, v)$. By definition $H' \in \mathcal{G}$ and thus $w(H')$ is well defined. We now use $w$ to define a weight function $w'$ on $H$. For every edge $e \in H$, let $w'(e) = w(e')$.

Now suppose $H$ has two distinct minimum weight perfect matchings, $M_1$ and $M_2$, with respect to $w'$. Then the symmetric difference of $M_1$ and $M_2$ is a collection of disjoint, even length, simple cycles, where the edges of the cycle alternate between the matchings $M_1$ and $M_2$. Since $M_1$ and $M_2$ are distinct, there is at least one cycle. Let $C = (v_1, v_2, \ldots, v_{2k}, v_1)$ be one such cycle. Let $e_i = (v_i, v_{(i+1) \pmod{2k}})$ for $i \in [2k]$. Without loss of generality assume, $v_1 \in L$ and the edge $e_1$ is in $M_1$. Therefore if $i$ is odd (resp. even), then $e_i \in M_1$ (resp $e_i \in M_2$) and $e'_i$ is directed from $L$ to $R$ (resp from $R$ to $L$). Thus $w'(e_{2i-1}) = w(e'_{2i-1})$ and $w'(e_{2i}) = -w(e'_{2i})$ for $i \in [k]$, due to the skew-symmetry of $w$.

The weight of the restriction of $M_1$ to $C$, $w'(M_1 \cap C) = \sum_{i=1}^{k} w'(e_{2i-1})$. Similarly $w'(M_2 \cap C) = \sum_{i=1}^{k} w'(e_{2i})$. Now,

$$
\begin{aligned}
w'(M_1 \cap C) - w'(M_2 \cap C) &= \sum_{i=1}^{k} w'(e_{2i-1}) - \sum_{i=1}^{k} w'(e_{2i}) \\
&= \sum_{i=1}^{k} w(e'_{2i-1}) + \sum_{i=1}^{k} w(e'_{2i}) = \sum_{i=1}^{2k} w(e'_i) \\
&\neq 0.
\end{aligned}
$$

Therefore either $M_1 \cap C$ or $M_2 \cap C$ has higher weight with respect to $w'$. Without loss of generality assume it is $M_2$. Thus we get a perfect matching $M' = M_2 \setminus (M_2 \cap C) \cup (M_1 \cap C)$ in $H$ of lesser weight, which is a contradiction.  □

## 4. Piecewise straight line embedding of a planar graph

In this section we give a log-space algorithm to compute a piecewise straight line embedding of a planar graph. All graphs considered in this section are undirected, unless otherwise specified.

**Definition 2.** For $p_i \in \mathbb{R}^2$, $(p_1, \ldots, p_{k+1})$ is said to be a piecewise straight line segment, if there is a straight line segment connecting $p_i$ with $p_{i+1}$ for every $i \in [k]$.

**Definition 3.** For $k \geq 1$, a $k$-piecewise straight line embedding of a graph $G = (V, E)$ is a function $f : V \to \mathbb{R}^2$ and a collection of $(k - 1)$ functions $g_i : E \to \mathbb{R}^2$ for $i \in [k - 1]$, such that every edge $e = (u, v) \in E$ is a piecewise straight line segment, $(f(u), g_1(e), \ldots, g_{l_e-1}(e), f(v))$ for some $l_e \leq k$ and no two embedded edges intersect except possibly at the end points.

**Theorem 7.** *Given a combinatorial embedding of a planar graph $G$, there is a log-space algorithm that computes a 4-piecewise straight line embedding of $G$.*

We will give an embedding of $G$ in the first quadrant of the coordinate plane. We first use Reingold's undirected reachability algorithm [13] to compute a spanning tree $T$ of $G$ rooted at some vertex $r$ in $G$. (See Fig. 1.) Now from $G$ we create a new graph $G_T$ by "cutting" every non-tree edge $e = (u, v)$ in two, to create two edges $(u, w_e^u)$ and $(v, w_e^v)$; thus $G_T$ is a tree and the newly introduced vertices $w_e^u$ and $w_e^v$ are leaves. We then give a straight line embedding of $G_T$ in the first quadrant of the two-dimensional Cartesian coordinate system (we shall just refer to it as the coordinate system from now on), such that the new leaves $w_e^u$ and $w_e^v$ (that is, the vertices that were introduced when cutting every non-tree edge $e$) lie on a circle centered at the origin and containing $G_T$. Next we reconnect the split edges appropriately to avoid intersections. Below we give a more formal description of the algorithm.

We create $G_T = (V_T, E_T)$ from $G$ as follows. For each edge $e = (u, v)$ in $E \setminus T$, we introduce two new vertices $w_e^u$ and $w_e^v$. Now replace $e$ with the edges $(u, w_e^u)$ and $(v, w_e^v)$. Denote the newly introduced set of vertices and edges as
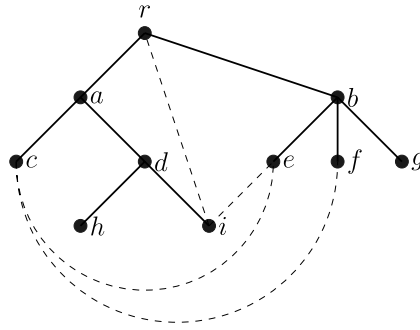
**Fig. 1.** Example of a graph $G$ containing a spanning tree $T$ (solid edges) rooted at $r$. The dashed edges are the non-tree edges.

$V'_T$ and $E'_T$. Thus $V_T = V \cup V'_T$ and $E_T = T \cup E'_T$. Note that $G_T$ is a tree and every vertex in $V'_T$ is a leaf. We shall think of $G_T$ as a tree rooted at $r$ as well. Next we define the height function, $h$, for every vertex in $G_T$. For the root node $h(r) = 0$. For every vertex $v \neq r$ in $V$, $h(v) = h(p) + 1$, where $p$ is the parent node of $v$ in $G_T$ and for every vertex $v$ in $V'_T$, $h(v) = \max\{h(u) + 1 : u \in V\}$. Define $h(G_T) = \max\{h(v) : v \in V_T\}$. For a vertex $v$, let $A(v)$ be the set of leaves $u$ in $G_T$, such that $u$ is not present in the subtree rooted at $v$ and the path from $u$ to $r$ lies to the left of the path from $v$ to $r$. Let $L$ be the set of leaves in $G_T$. Then $\theta(v) = \frac{|A(v)|}{|L|} \frac{\pi}{2}$.

The coordinates of a vertex $v$ in our embedding are

$$F(v) = \big(h(v)\cos(\theta(v)), h(v)\sin(\theta(v))\big).$$

For every edge $e = (u, v) \in E_T$ draw a straight line segment between $F(u)$ and $F(v)$ to represent the edge. We shall denote the embedding of this edge (line segment) by $F(e)$. Note that the circle with radius $h(G_T)$ contains the entire embedded graph $G_T$, and that the vertices $v$ in $V'_T$ lie on this circle.

Next we compare the sets $A(u)$ and $A(v)$ for two vertices $u$ and $v$.

**Lemma 8.** *Let $u$ and $v$ be two distinct vertices in $G$. (a) If $u$ is an ancestor of $v$ then $A(u) \subseteq A(v)$. (b) If $u$ lies to the left of $v$, then $A(u) \subsetneq A(v)$ (and hence for any descendant $w$ of $u$, $A(w) \subsetneq A(v)$).*

**Proof.** (a) follows from the observation that any vertex to the left of a node also lies to the left of any of its descendants. Similarly, if $u$ lies to the left of $v$, then any node to the left of $u$ also lies to the left of $v$. The containment is proper, since leaves in the subtree rooted at $u$ are contained in $A(v)$ but not in $A(u)$. Also if $w$ is a descendant of $u$, then $w$ lies to the left of $v$ and hence $A(w) \subsetneq A(v)$. This proves (b).  □

In Lemma 9 we show that distinct vertices get mapped to distinct coordinates by $F$. In Lemma 10 we prove that no two edges of $G_T$ intersect at an intermediate point.

**Lemma 9.** *Let $u$, $v$ be two vertices in $G_T$. Then $u = v$ if and only if $F(u) = F(v)$.*

**Proof.** Let $u$ and $v$ be two distinct vertices. If $h(u) \neq h(v)$ then $F(u) \neq F(v)$ since they lie in different concentric cycles around the origin by definition of $F$. If $h(u) = h(v)$, then it follows from Lemma 8.  □

**Lemma 10.** *Let $e_1$ and $e_2$ be two edges in $G_T$. Then $F(e_1)$ and $F(e_2)$ do not intersect except possibly at end points.*

**Proof.** Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ such that $u_i$ is the parent of $v_i$. If $u_1 = u_2$ then since $v_1 \neq v_2$, $e_1$ and $e_2$ do not intersect non-trivially. Also if $v_1$ is an ancestor of $u_2$ then $v_1$ is an ancestor of $v_2$ as well and therefore they cannot intersect since they lie in concentric circles of different length around the origin.

We now consider the case when $u_1$ is not an ancestor or descendant of $u_2$. Without loss of generality assume $u_1$ is to the left of $u_2$, which implies that $A(u_1) \subsetneq A(u_2)$. From Lemma 8 we get $\theta(u_1) \leqslant \theta(v_1) < \theta(u_2) \leqslant \theta(v_2)$. Therefore the line segments $(F(u_1), F(v_1))$ and $(F(u_2), F(v_2))$ do not intersect.  □

Next we rejoin the split edges to get back the original graph. After joining, a split edge would be embedded as a piecewise straight line as we describe below. (See Fig. 2.) Recall that precisely the non-tree edges in $G$ are the edges that were split.

Suppose $e = (u, v)$ was a non-tree edge in $G$. Then $e$ was replaced by the edges $(u, w^u_e)$ and $(v, w^v_e)$ by the introduction of two new vertices $w^u_e$ and $w^v_e$. We remove the vertices $w^u_e$ and $w^v_e$ and the edges $(u, w^u_e)$ and $(v, w^v_e)$ and draw the
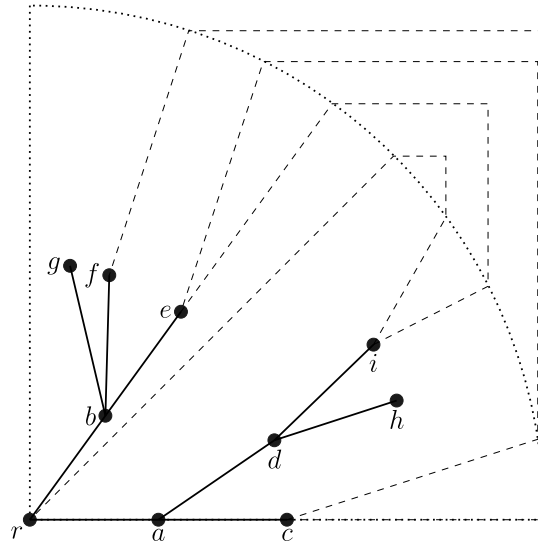
**Fig. 2.** Piecewise straight line embedding of $G$.

piecewise straight line segment

$$\left(F(u), F\left(w_e^u\right), \max\left\{F\left(w_e^u\right), F\left(w_e^v\right)\right\}, F\left(w_e^v\right), F(v)\right)$$

to represent edge $e$ (where the max function is defined as $\max\{(a_1, b_1), (a_2, b_2)\} \triangleq (\max\{a_1, a_2\}, \max\{b_1, b_2\})$).

We shall denote the embedding of this edge (piecewise line segment) by $F(e)$. In Lemma 11 we show that non-tree edges do not intersect non-trivially, to complete the proof of Theorem 7.

**Lemma 11.** *Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be two non-tree edges in $G$. Then the edges $F(e_1)$ and $F(e_2)$ do not intersect non-trivially.*

**Proof.** We only need to show that the piecewise line segments

– $(F(w_{e_1}^{u_1}), \max\{F(w_{e_1}^{u_1}), F(w_{e_1}^{v_1})\}, F(w_{e_1}^{v_1}))$ and
– $(F(w_{e_2}^{u_2}), \max\{F(w_{e_2}^{u_2}), F(w_{e_2}^{v_2})\}, F(w_{e_2}^{v_2}))$

do not intersect.

**Case 1** *(One end point of $e_1$ and $e_2$ is common).* Without loss of generality assume $u_1 = u_2 = u$ (say) and $\theta(v_1) \leqslant \theta(v_2)$. Thus in $G_T$, $w_{e_2}^u$ lies to the left of $w_{e_1}^u$ which implies $\theta(w_{e_2}^u) < \theta(w_{e_1}^u)$ by Lemma 8. Also $\theta(w_{e_1}^{v_1}) < \theta(w_{e_2}^{v_2})$ since $w_{e_1}^{v_1}$ and $w_{e_2}^{v_2}$ are children of $v_1$ and $v_2$ respectively. This shows the lemma for Case 1.

**Case 2** *(All end points of $e_1$ and $e_2$ are distinct).* Without loss of generality assume $\theta(u_1) \leqslant \theta(u_2)$ and $\theta(u_i) \leqslant \theta(v_i)$ for $i \in \{1, 2\}$. Since $e_1$ and $e_2$ cannot intersect, therefore if $\theta(v_1) \geqslant \theta(u_2)$, then $\theta(u_1) \leqslant \theta(v_2) \leqslant \theta(v_1)$, and if $\theta(v_1) < \theta(u_2)$, then either $\theta(v_2) \geqslant \theta(v_1)$. This implies that either $\theta(w_{e_1}^{u_1}) < \theta(w_{e_2}^{u_2}) < \theta(w_{e_2}^{v_2}) < \theta(w_{e_1}^{v_1})$ or $\theta(w_{e_1}^{u_1}) < \theta(w_{e_1}^{v_1}) < \theta(w_{e_2}^{u_2}) < \theta(w_{e_2}^{v_2})$. Hence the lemma holds for this case too.  □

Note that the coordinates that we assign are real numbers and need not be computable in log-space. To take care of this problem we can "inflate" the entire mapping by multiplying each coordinate with a suitable large number (say $|V|^5$) and then taking the floor of each point to get an integral embedding. Also it has been shown that good enough approximations of the functions $\sin(\theta)$ and $\cos(\theta)$ and the constant $\pi$ can be computed efficiently [14,15].

As an application of Theorem 6 and 7, we get a simpler proof of the following result from Datta, Kulkarni and Roy [8].

**Corollary 12.** *(See [8].) There is a log-space computable weight function, with respect to which there is a unique minimum weight perfect matching in every bipartite, planar, undirected graph.*

Lemma 3 gives a weight function for the class of directed planar graphs that is skew-symmetric. Then we apply Theorem 6 to get the above corollary.

## 5. Final remarks

It is clear that there are many other weight functions that will work. In fact any "nice" solution to the differential equation $(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) = 1$ will yield isolating weight functions. In particular, setting $P(x, y) = \frac{-y}{2}$ and $Q(x, y) = \frac{x}{2}$ to the left hand side of Green's theorem yields the weight function $w(e) = (x_u y_v - x_v y_u)$ which is isolating.

One can easily verify that the weight function we give here is a true extension of the following weight function prescribed in [7] for isolating paths in grid graphs: east and west edges are given 0 weight, a north edge at $((i, j), (i, j + 1))$ is given a weight $i$, and a south edge at $((i, j), (i, j - 1))$ is given a weight $-i$. However, if we apply our theorem for the case of isolating matching in grid graphs, we get a different (slightly simpler) weight function than the one prescribed in [8]. We believe that this paper better explains the reason behind why these weight functions work.

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