Theory and Practice of Constraint Satisfaction
CSCE 990-05, Fall 1999
www.cse.unl.edu/~choueiry/CSE990-05/

Title:	The complexity of some polynomial consistency algorithms for constraint satisfaction problems
Authors:	A.K. Mackworth A.K. and E.C. Freuder
Journal:	Artificial Intelligence, Vol 25
Pages:	65–74
Year:	1985

Thrashing

exploring non-promising subtrees
and rediscovering the same inconsistent
combinations over and over again

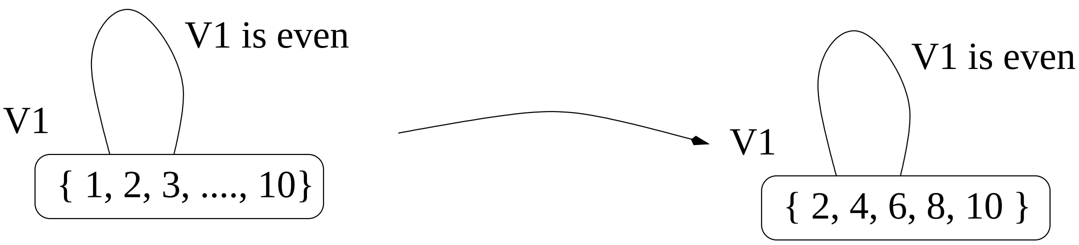
Malady of backtrack search

Consistency checking

eliminates inconsistent combinations
prior to, or during, search

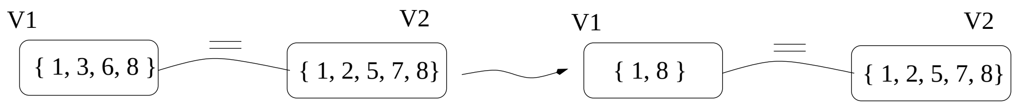
- Node consistency
- Arc consistency (AC-1, AC-2, AC-3)
- Path consistency (PC-1, PC-2)

Node consistency 1 pass

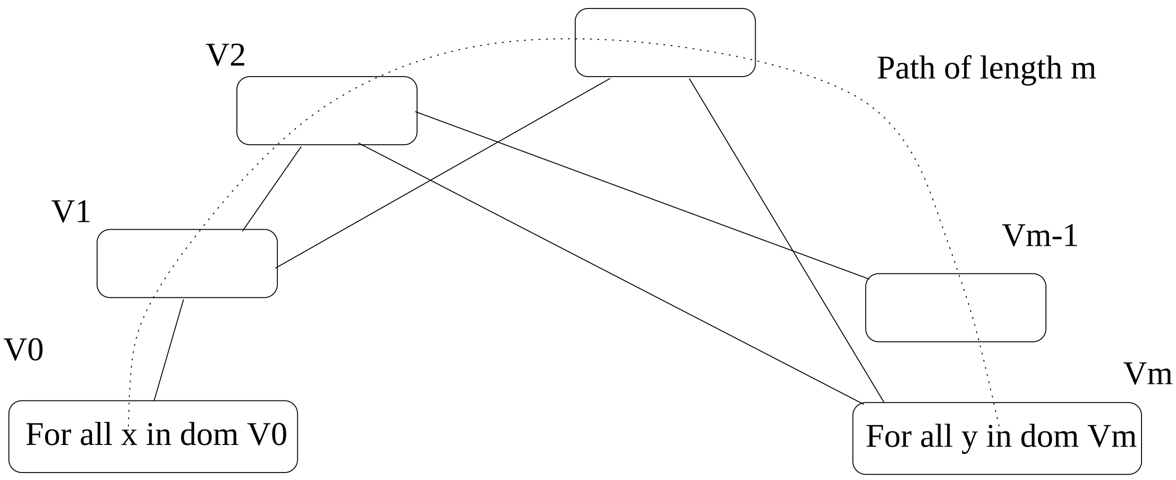


Arc consistency

iterates over arcs until quiescence



Path consistency for path of length m



Problem size parameters:

$a =$ (max) size of a domain

$d_k =$ degree of V_k ($\leq n-1$)

$n =$ number of variables

$e =$ number of edges $\in [(n-1), n(n-1)/2]$

Worst-case complexity $\begin{cases} \text{time} \\ \text{space} \end{cases}$

as a function of the input parameters

$f(n)$ is $\mathcal{O}(g(n))$ means that $f(n) \leq c.g(n)$
f grows as g or slower

$f(n)$ is $\Omega(h(n))$ means that $f(n) \geq c.h(n)$

[Papadimitriou 94, Sect 1.1]

Complexity of NC

Include NC algorithm from Mackworth and Freuder'85

For each variable, we check a values

We have n variables, we do $n.a$ checks

NC is $\mathcal{O}(a.n)$

AC-1 does not update Q, the queue of arcs

AC-2 iterates over arcs connected to at least one node whose domains has been modified. Nodes are ordered.

AC-3 same as AC-2, nodes are not ordered

Complexity of AC-1

Include AC-2 algorithm from Mackworth and Freuder'85

Q is not modified, and $|Q| = 2e$

4—9 repeats at most $n.a$ times

Each iteration has $|Q| = 2e$ calls to `REVISE`

`REVISE` requires at most a^2 checks of P_{ij}

$\longrightarrow$ AC-1 is $\mathcal{O}(a^3.n.e)$

Complexity of AC-3

Include AC-3 algorithm from Mackworth and Freuder'85

First $|Q| = 2e$, then it grows and shrinks 4—8

Worst case:
$$\begin{cases} \text{1 element is deleted } D_{V_k} \text{ per iteration} \\ \text{none of the arcs added is in } Q \end{cases}$$

Line 7:	$a.(d_k - 1)$
Lines 4–8:	$2e + \sum_{k=1}^{n} a.(d_k - 1) = 2e + a.(2e - n)$
`REVISE`:	a^2 checks of P_{ij}

$\longrightarrow$ AC-3 is $\mathcal{O}(a^2(2e + a(2e - n)))$
Connected graph $\Rightarrow e \geq n - 1$, AC-3 is $\mathcal{O}(a^3 e)$

If AC-p, AC-3 is $\mathcal{O}(a^2 e) \Rightarrow$ AC-3 is $\Omega(a^2 e)$
Complete graph: $\mathcal{O}(a^3 n^2)$

Representing relations

Constraint, relation:

$$C_{V_k,V_l,V_m} = \{\,(x, y, z)\} \subseteq D_{V_k} \times D_{V_l} \times D_{V_m}$$

Binary constraint/relation:

$$C_{V_k,V_l} = \{\,(x, y)\} \subseteq D_{V_k} \times D_{V_l}$$

$$\begin{pmatrix} 1 & 0 & 1 & \ldots & 1 \\ 0 & \ldots & & & 1 \\ 1 & \ldots & & & 1 \end{pmatrix}$$

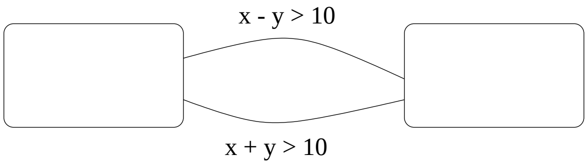
Operations on relations

- Intersection of relations

relations apply to the same two variables

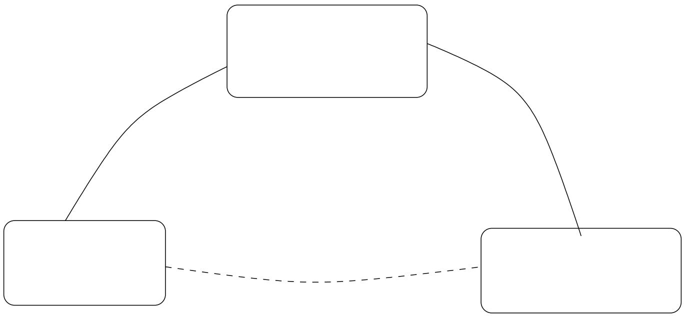
are replaced by a single tighter relation

by a logical `AND`



- Composition of relations

a simple (binary) matrix multiplication



Direct and induced relations

Minimal network

Given two values for two variables, if they are consistent, then they appear in at least one solution.

Theorem

In a complete graph, if every path of length 2 is consistent, the network is path consistent (and is minimal).

Bliek & Sam-Haroud (IJCAI 99):

- Theorem holds for triangulated graphs
- For convex constraints: PC has same pruning power on triangulated as on complete graphs
- For non-convex constraints: comparable performance of pruning shown by experimental evidence (not theoretic).

PC-1

Achieved by composition and intersection (of binary relations expressed as matrices) over all paths of length two.

PC-2

Improvement similar to (AC-1 $\rightarrow$ AC-3)

Complexity of PC-1

Include PC-1 algorithm from Mackworth and Freuder'85

Line 9: a^3
Lines 6–10: $n^3.a^3$
Line 3: at most n^2 relations$\times a^2$ elements

PC-1 is $\mathcal{O}(a^5n^5)$

PC-3 is $\mathcal{O}(a^5n^3)$ and $\Omega(a^3n^3)$

Mohr & Henderson (AIJ 86):

$$\begin{cases} \text{AC-3, } \mathcal{O}(a^3e) \longrightarrow \text{AC-4, } \mathcal{O}(a^2e) \\ \text{AC-4 is optimal.} \\ \\ \text{PC-2, } \mathcal{O}(a^5n^3) \longrightarrow \text{PC-3, } \mathcal{O}(a^3n^3) \\ \text{Open question: PC-3 optimal?} \end{cases}$$

Han & Lee (AIJ 88):

$$\begin{cases} \text{PC-3 is incorrect} \\ \text{PC-4, } \mathcal{O}(a^3n^3) \text{ space and time} \end{cases}$$

In practical applications:

$\surd$ AC-1, AC-2, etc.

$\times$ PC-1, PC-2, etc.