Title: Eliminating Interchangeable Values
in Constraint Satisfaction Problems

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Proc.: AAAI

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Advanced Constraint Processing

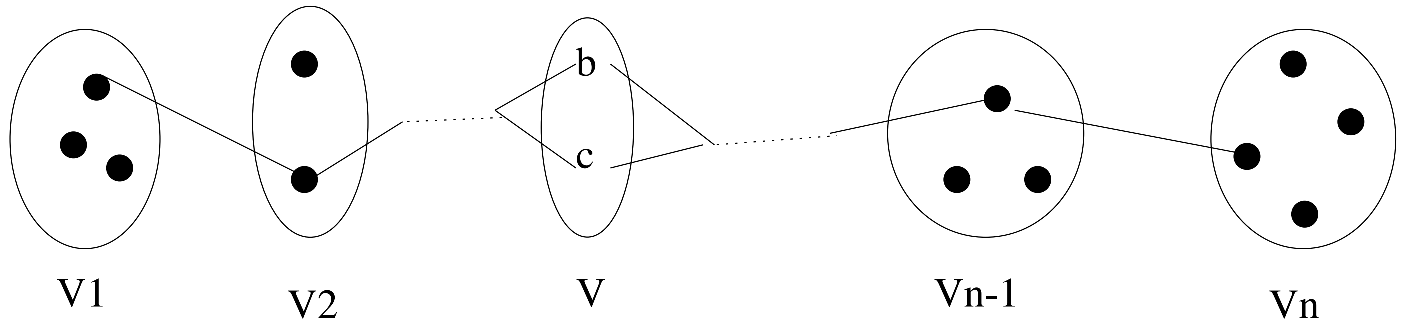
CSCE990-02, Fall 2010

`www.cse.unl.edu/~choueiry/F09-990-02/`

Fully interchangeable (FI) values (I)

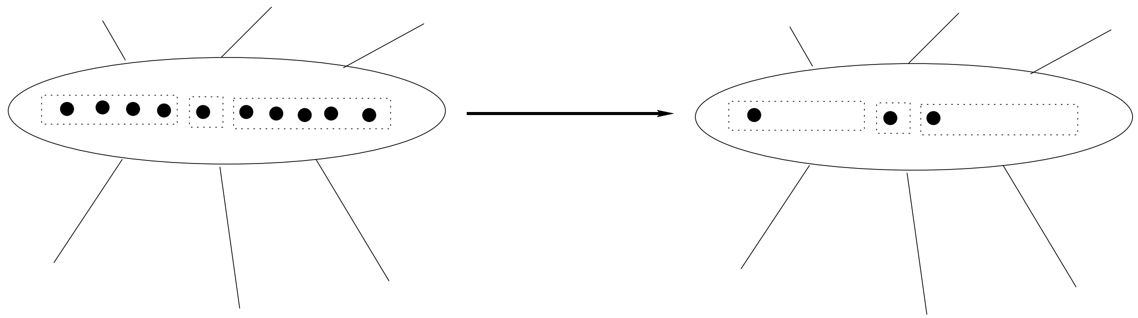
A value b for a CSP variable V is fully interchangeable with a value c for V iff:

1. Every solution to the CSP that contains b remains a solution when c is substituted for c
2. Every solution to the CSP that contains c remains a solution when b is substituted for c.



Full interchangeability (II)

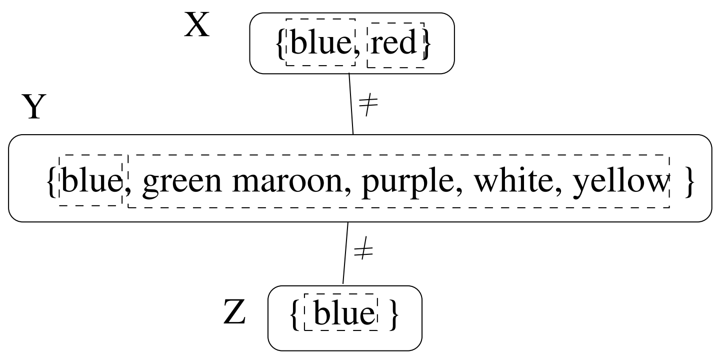
FI partition the domain of a CSP variable into sets of equivalent values



We can replace a set of FI values with a single representative of the set without effectively losing any solutions

Also, we need not retain equivalent values: solutions involving the representative can be transformed into solutions involving any other value in the set

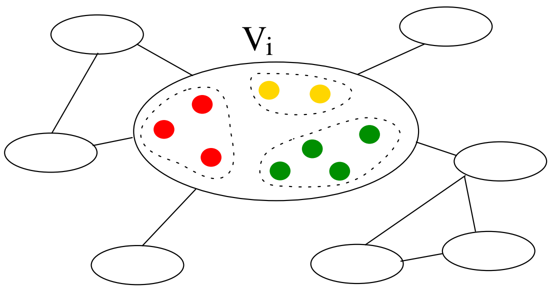
Examples

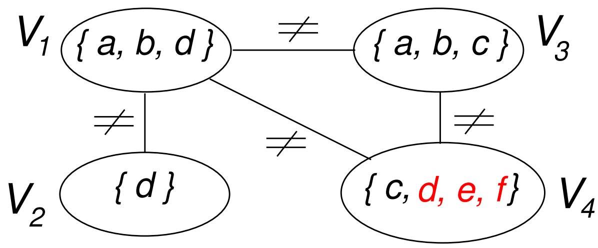


X← red

Y← green, maroon, purple, white, yellow

Z← blue





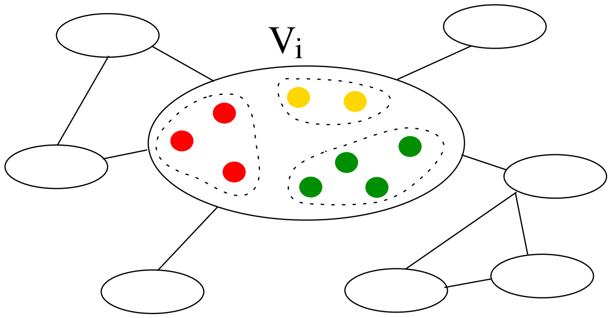
Advantages

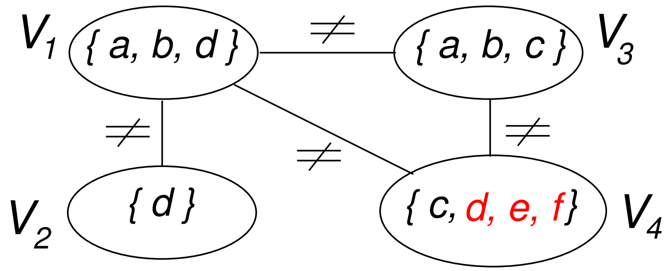
- Eliminating interchangeable values can prune a great deal of effort from a backtrack search tree.
- If we are seeking all solutions, interchangeability allows us to find a <u>family of similar solutions</u> and avoid duplication of effort
- Basis for explanation, concept formation and problem decomposition

Contents of paper

- Introduces interchangeability: symmetry among the values of a CSP variable
- 'Pure:' full interchangeability (FI)
- Approximation:
  - Neighborhood (NI), k-interchangeability (from global to **<u>local</u>**) *+algorithm*
  - Extended:
    - Substitutability, partial interchangeability (PI), subproblem interchangeability (from strong to **<u>weak</u>**)
    - Meta interchangeability
    - Dynamic interchangeability
    - Functional interchangeability

Full interchangeability (FI)



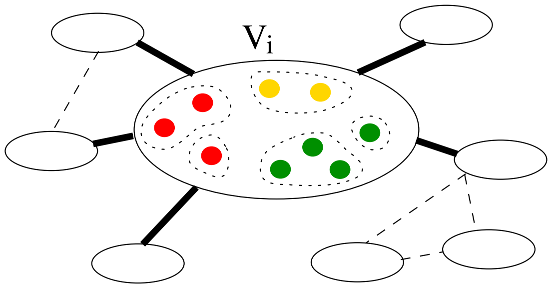


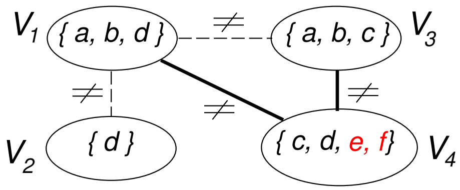
Neighborhood interchangeability (NI)

Two values, a and b for a CSP variable V are NI iff for every C on V:

$$\{i \mid (a,i) \text{ satisfies } C\} = \{i \mid (b,i) \text{ satisfies } C\}$$

NI is restricted to $\texttt{Neigh}(V)$





Theorem 1: NI $\Rightarrow$ FI

Neighborhood interchangeability is a sufficient, but not a necessary condition for full interchangeability

Proof:

If two values are NI, there is no way they could fail to be interchangeable in any complete solution since there is not constraint that one could satisfy and not the other

But, FI values may not be NI: previous example, insoluble problem

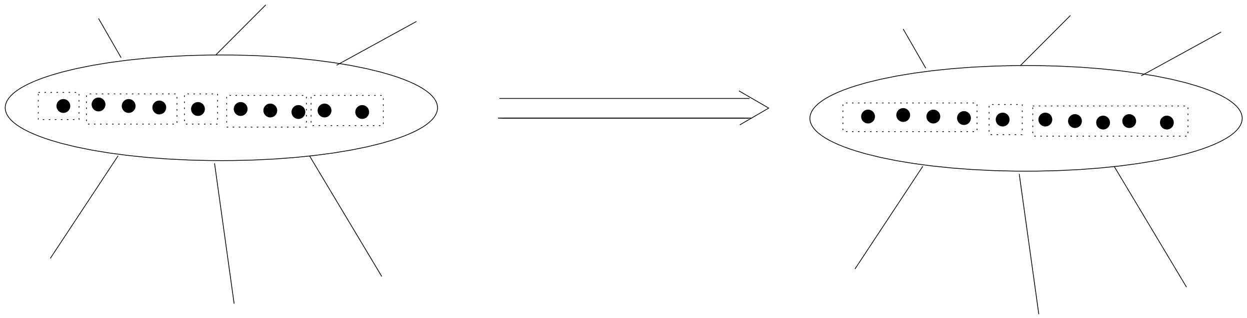
NI as an approximation for FI

NI considers only constraints in neighborhood of the variable

FI considers the effects of all constraints in the problem on the variable

NI $\Rightarrow$ FI

For each NI-set s_i there is an FI-set s_j such that $s_i \subseteq s_j$



Finding the NI-partition only needs polynomial time

Finding FI-partition may require finding all solutions to problem..

Computing NI: discrimination tree (DT)

Algorithm DT for V_i $(D_{V_i}, \texttt{Neigh}(V_i))$

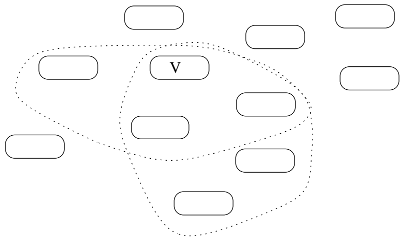
Create the root of the discrimination tree
Repeat for each value $v_i \in D_{V_i}$:
  Repeat for each variable $V_j \in \texttt{Neigh}(V_i)$:
    Repeat for each $v_j \in D_{V_j}$ consistent with v_i for V_i:
      Move to if present, construct and move to if not,
      a child node in the tree corresponding to '$V_j = v_j$'.
  Add '$V_i, \{v_i\}$' to annotation of the node (or root),
  Go back to the root of the discrimination tree.

The annotations in the tree give the NI-sets for V_i

Example: on board.
Time complexity:...
Space complexity:...

K-interchangeability

Two values, a and b for a CSP variable V are k-interchangeable iff a and b are fully interchangeable in **every** subproblem of the CSP induced by V and $(k-1)$ other variables



- NI = x-interchangeability, $x =$?
- FI = x-interchangeability, $x =$?
- When $k < n$, k-interchangeability is local
- **Theorem 2**: $\forall i < j$, i-interchangeability $\Rightarrow$ j-inter.
- Algorithm: generalization of DT, requires solving subproblems of size $k-1$
- Complexity: $O(n^k d^k)$, $n =$#variables and $d =$ domain size

Interchangeability types

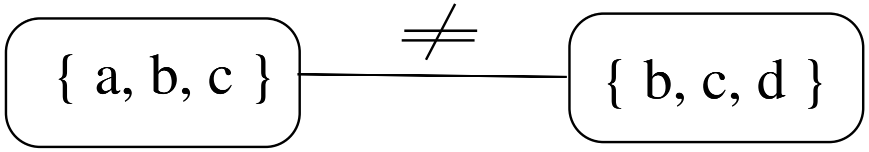
From local to global: NI $\rightarrow$ k-interchangeability $\rightarrow$ FI

Extended interchangeability:

- Weak Interchangeability: valid in some solutions to the CSP, not all
  - Substitutability
  - Partial interchangeability
  - Subproblem interchangeability
- Meta-interchangeability
- Dynamic interchangeability
- Functional interchangeability

Substitutability: One-way interchangeability

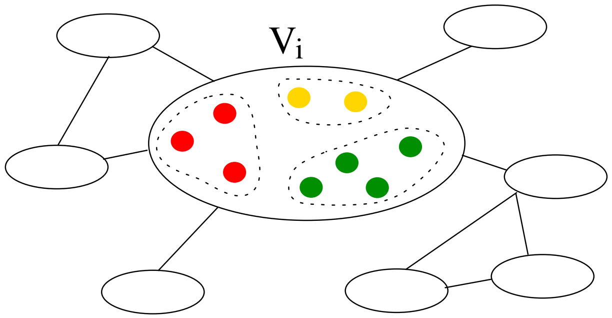
Given two values, a and b for a CSP variable V, a is substitutable for b iff substituting a in any solution involving b yields a solution

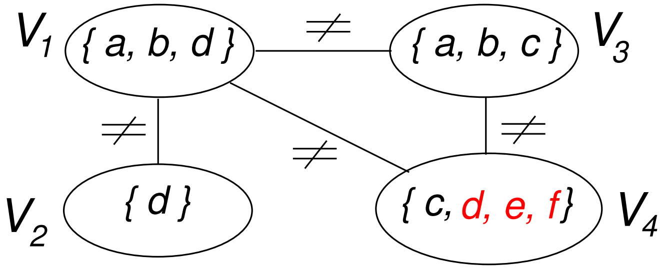


Note:

- a is substitutable for $b \not\Rightarrow b$ is substitutable for a
- One can define neighborhood substitutability

Substitutability (Sub)





d, e, f SUB c for $V_4 \Rightarrow$

d, e, f can replace c in all solutions with $V_4 = c$

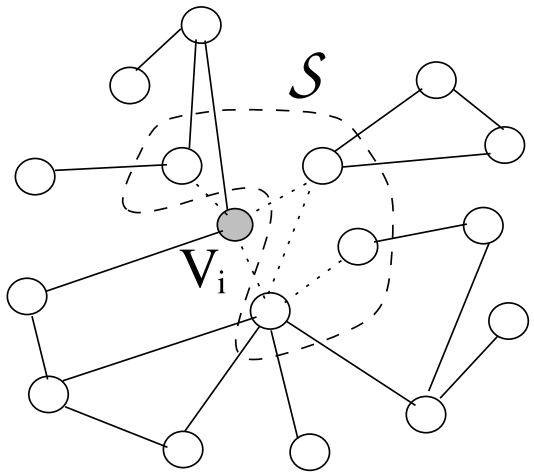
$\rightarrow$ convert solutions to reinforce preferences

Partial Interchangeability (PI)

The values for variables may differ among themselves, but be fully interchangeable with respect to the rest of the world

Idea: define a boundary to confine change

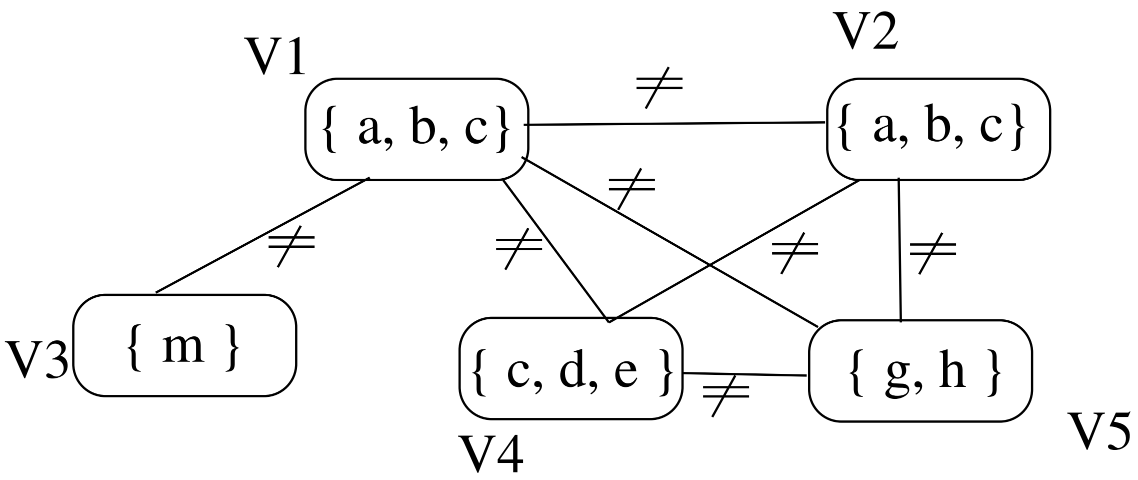
Two values are partially interchangeable with respect to a subset S of variables, iff any solution involving one implies a solution involving the other with possibly different values for variables in S



When S is empty, what do we have??

PI: example

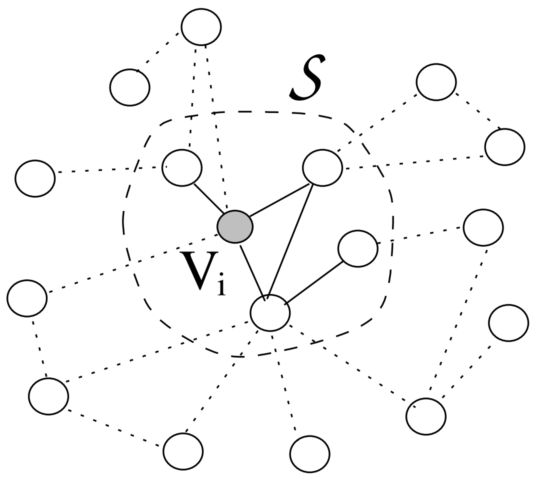
PI values for $V1$ given the boundary $S = \{V2\}$



One can define neighborhood partial interchangeability (NPI)

Subproblem interchangeability

Two values are subproblem interchangeable with respect to a subset of variables S iff they are fully interchangeable with respect to the solution of the subproblem of the CSP induced by S

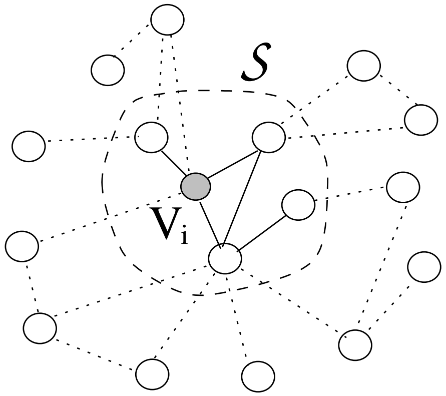


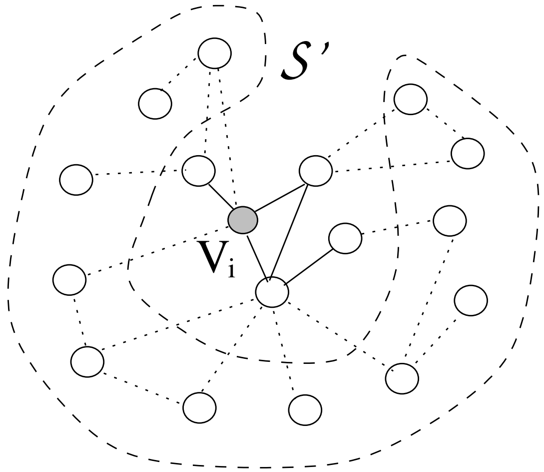
Subproblem interchangeability vs. PI

Theorem 5: Given S, and $S' = \mathcal{V} - S$ *bug in paper*

Subproblem interchangeability WRT $S \not\Rightarrow$ PI WRT S'

PI WRT $S' \Rightarrow$ Subproblem interchangeability WRT S





Proof: the key idea is that a solution to a subproblem may fail to appear as a portion of any solution to the complete problem

Meta-interchangeability

By grouping:

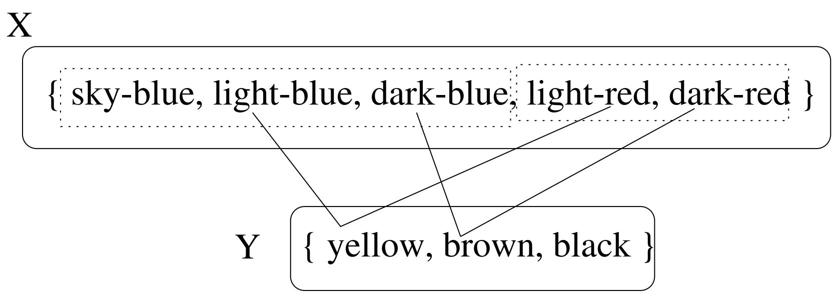
- variables into meta-variables or
- values into meta-values

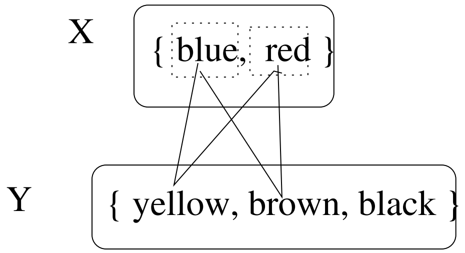
we can introduce interchangeability into a higher level representation of the original CSP.

Variables → meta-variables

Values → meta-values

CSP → meta-CSP





Yellow and brown for Y become fully interchangeable

Dynamic interchangeability

Interchangeability can be recalculated after choices are made for variables values during backtrack search

Functional interchangeability

Notation: $S_{v|V}$ be the set of solutions with $V \leftarrow v$.

Two values a for variable V and b for variable W are functionally interchangeable iff there exist functions f_a and f_b such that:

$$f_a(S_{a|V}) = S_{b|W} \text{ and } f_b(S_{b|W}) = S_{a|V}$$

Special case: $V = W$

In general, find all solutions, define your mappings :—(

Isomorphic interchangeability

Two values a and b for a CSP variable are isomorphically interchangeable iff there exists a 1-1 function f such that:

1. $b = f(a)$
2. for any solution S involving a, $\{f(v)|v \in S\}$ is a solution
3. for any solution S involving b, $\{f^{-}1^{(}b)|v \in S\}$ is a solution

Example: 8-queens

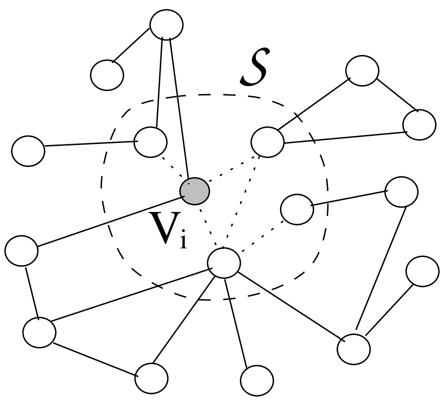
- For each row, values in col are isomorphically inter.
  $f : i \rightarrow (9 - i)$
- Also, $f : (i, j) \rightarrow (9 - i, 9 - j)$ *double-check*

Recent developments

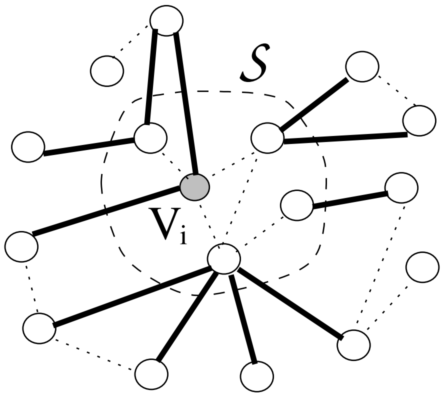
- Neighborhood Partial Interchangeability (NPI)
  Using the Joint Discrimination Tree [Choueiry & Noubir 98]

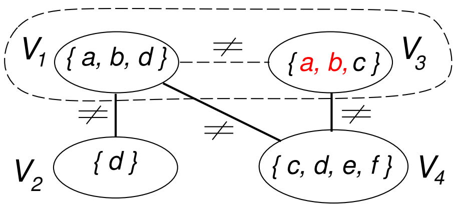
- Dynamic bundling (DNPI in search) [Choueiry & Davis 01]

Partial (PI) [Freuder'91]



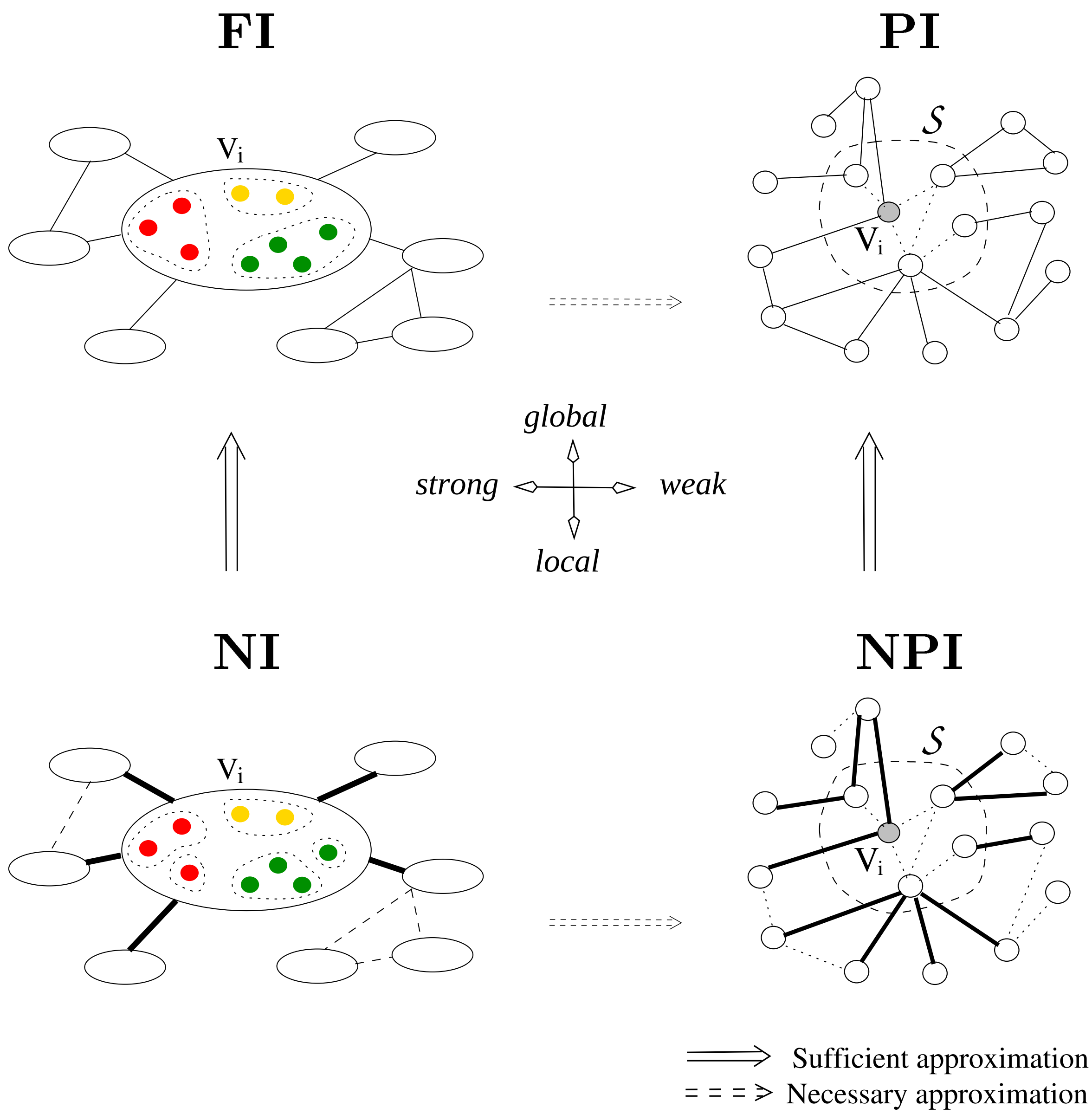
Neighborhood partial(NPI)





Neighborhood interchangeability	$\longrightarrow$	NI
Neighborhood partial interchang.	$\longrightarrow$	NPI
Substitutability	$\longrightarrow$	SUB
Neighborhood of $\mathcal{S}$	$\longrightarrow$	$\texttt{Neigh}(\mathcal{S})$

Control parameters $\begin{cases} \text{choice of } \mathcal{S} \\ \text{neighborhood of } \mathcal{S} \end{cases}$



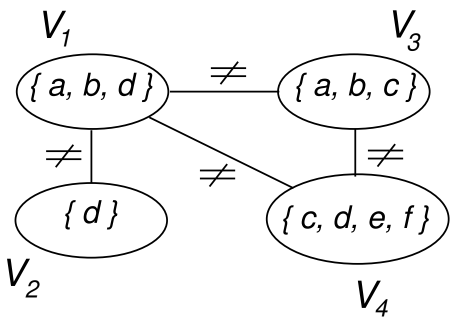
FI, PI (likely) intractable & PI is cheaper

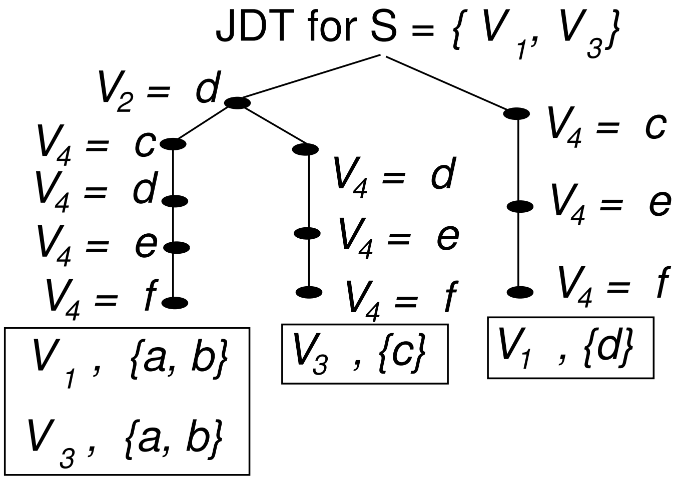
NI, NPI polynomial & NPI is cheaper

Joint discrimination tree $\begin{cases} \mathcal{S} \\ \texttt{Neigh}(\mathcal{S}) \\ \rightarrow \text{tree} \end{cases}$

$\forall\ V_i \in \mathcal{S}$,

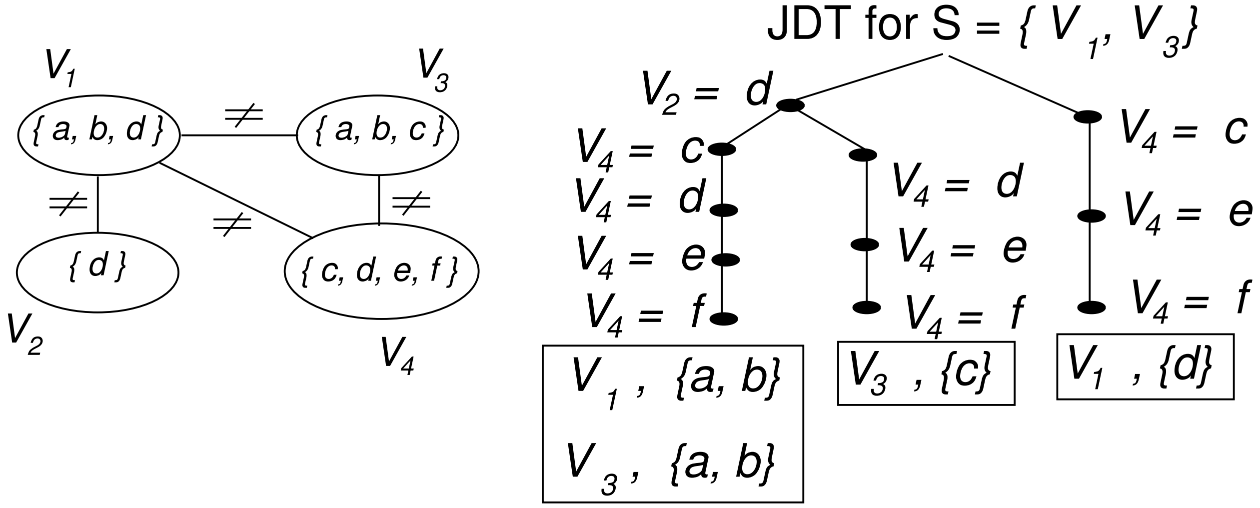
- $\forall\ v_i \in D_{V_i}$
  - $\forall\ V_j \in \texttt{N}eigh(\mathcal{S})$
    - $\forall\ v_j \in D_{V_j}$
    - v_i, v_j consistent? ...
  - Annotate node with(V_i, v_i)
- Return to root





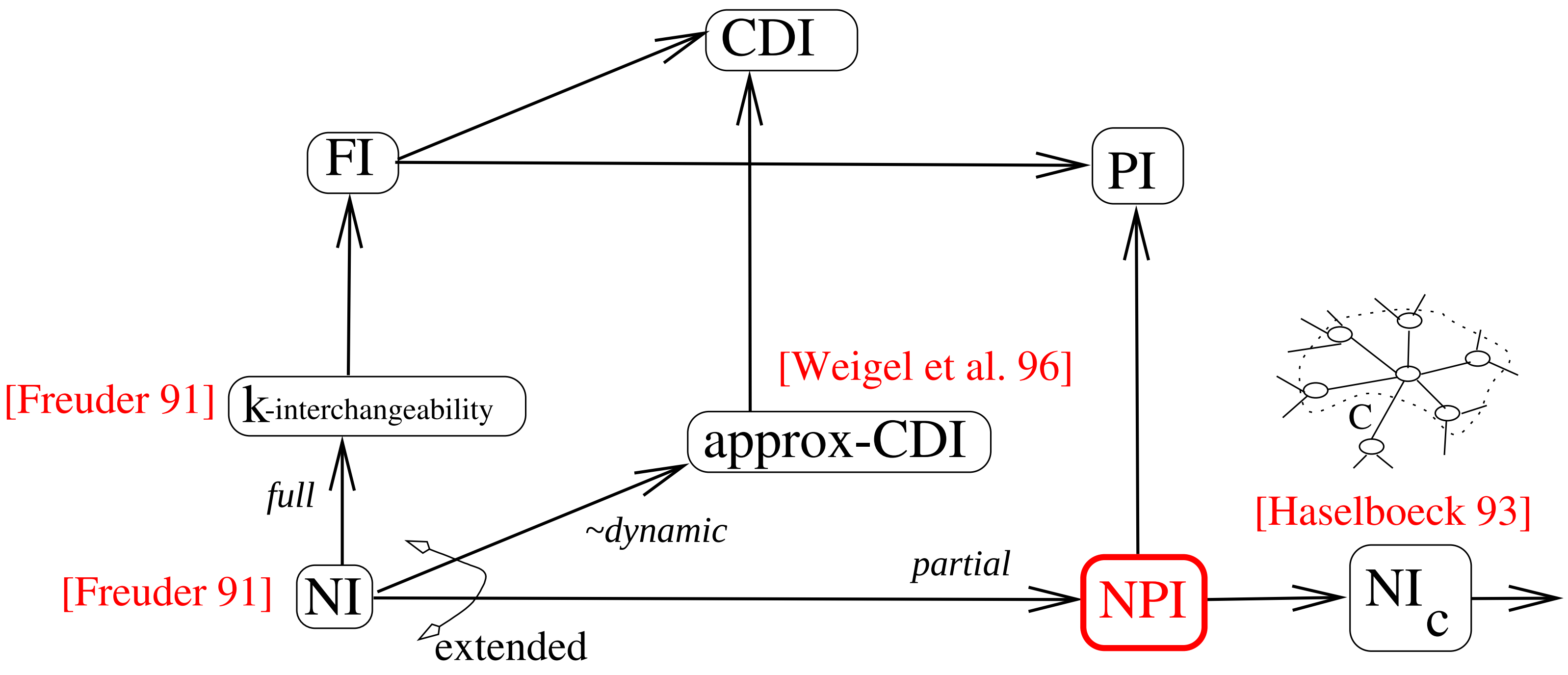
Complexity $O(s(n-s)d^2)$ $\begin{cases} s = |\mathcal{S}| \\ n = |\mathcal{V}| \\ d = \text{max domain size} \end{cases}$

Joint discrimination tree (cont')



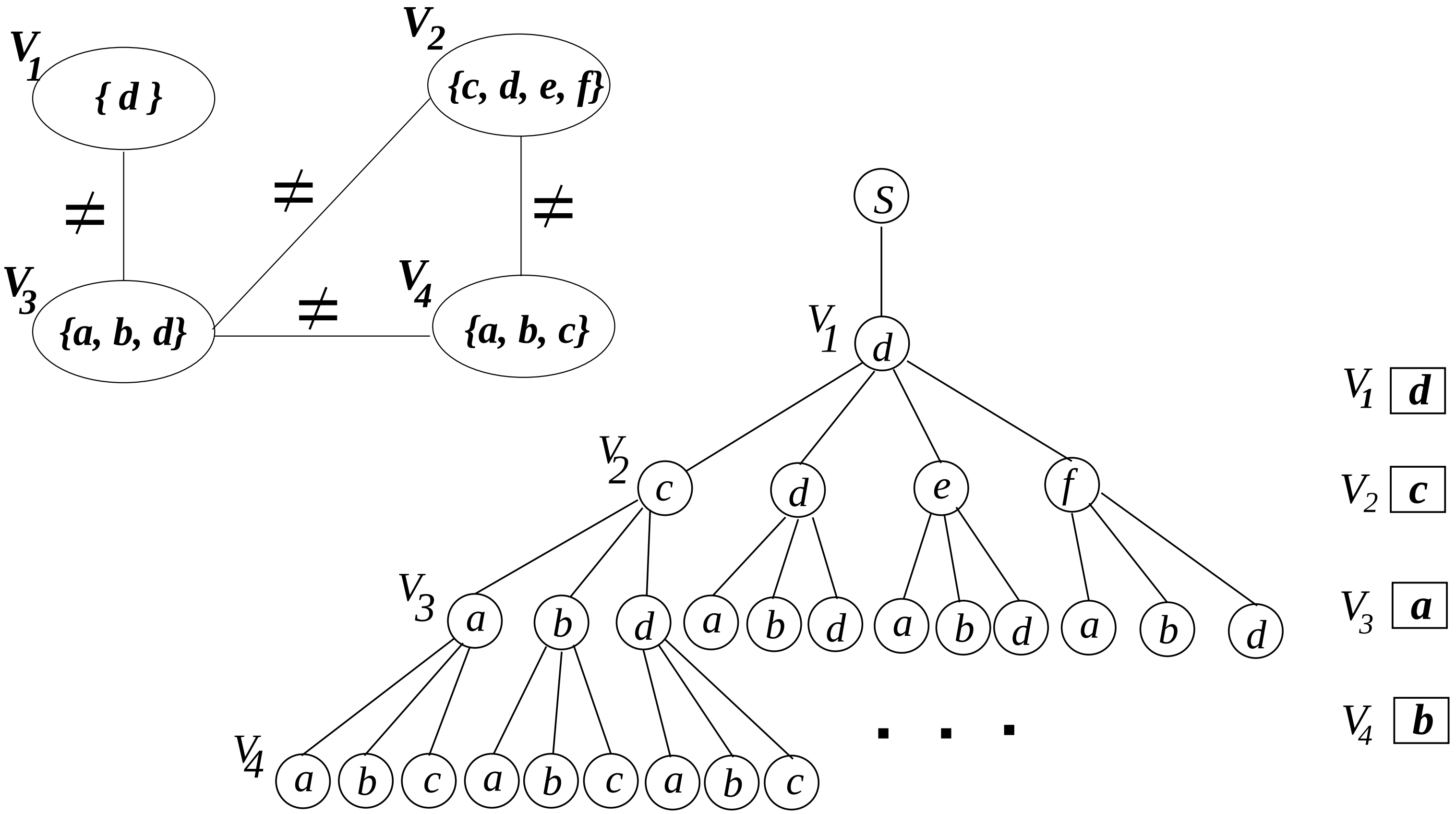
- Annotations partition domains of variables in $\mathcal{S}$
- Non-singleton sets in partition $\rightarrow$ NPI
- NPI sets for all variables in $\mathcal{S}$
- For one variable: $O(s(n-s)d^2) \rightarrow O((n-s)d^2)$

Lattice of some interchangeability relations

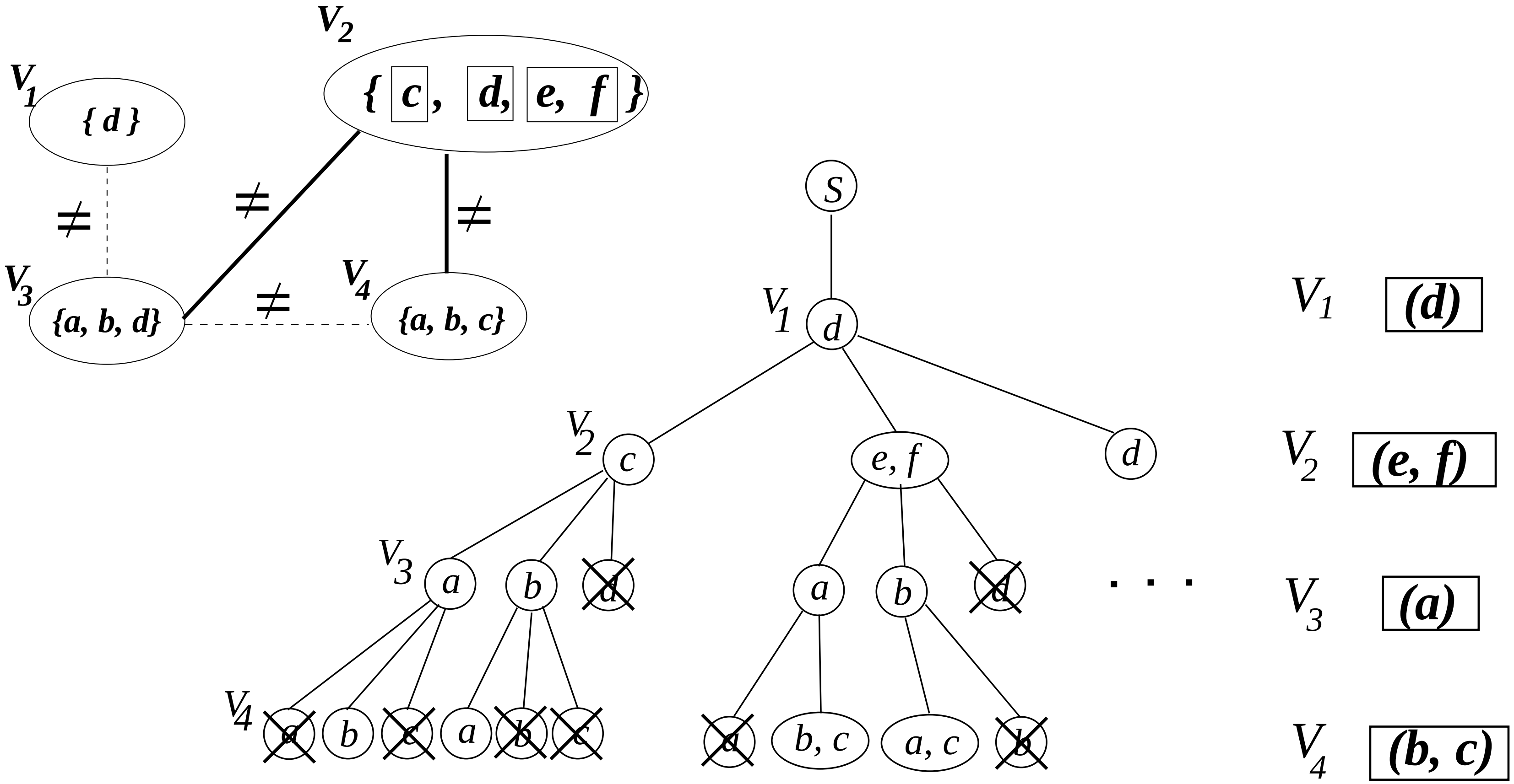


→ ‘Tunable’

Context: CSP and backtrack search



Search with bundling



Advantages of search with bundling

- Reduces search space
- Creates bundled solutions
  - Compact representation of the solution space
  - Each bundle is a set of robust solutions

V_1 (d)

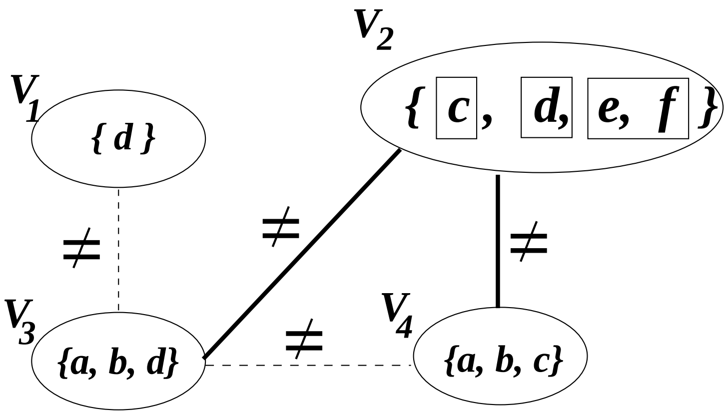
V_2 (e, f)

V_3 (a)

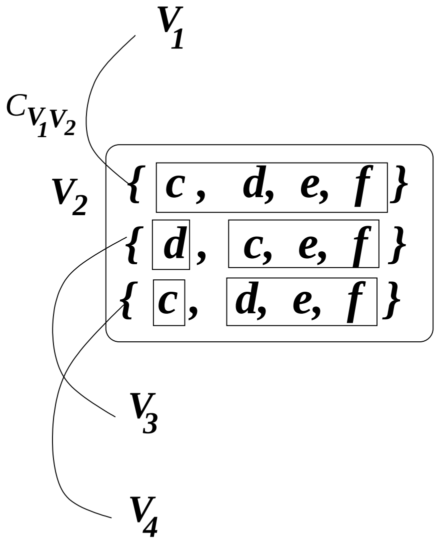
V_4 (b, c)

Static bundling

NI [Benson & Freuder 92]



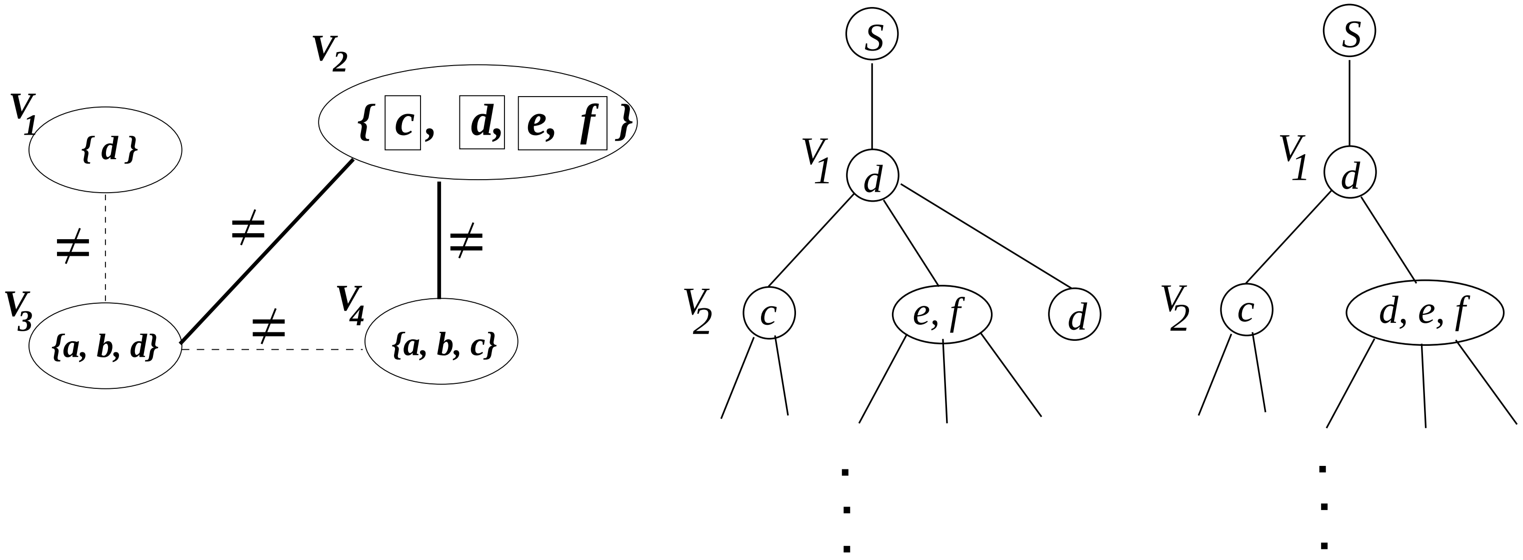
NI$_C$ [Haselböck 93]



– Before search, find interchangeable values

– Store interchangeable values for duration of search

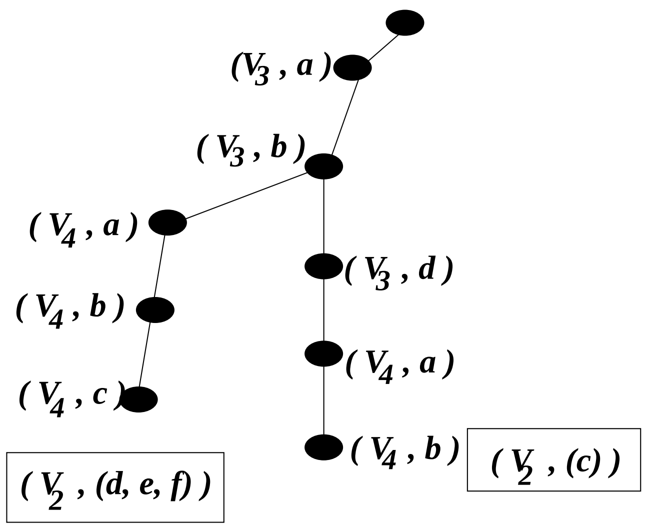
Dynamic bundling, during search



- As search proceeds, 'commitments' are made and more values become interchangeable [Freuder 91]

- Compute interchangeability sets considering constraints with future variables (NPI) using the joint discrimination tree (JDT) [Choueiry & Noubir 98]

Dynamic bundling and forward checking



- Recompute NPI sets, dynamic NPI (DNPI)
  Since the JDT requires the same constraint checks as forward checking (FC), use the JDT for FC [Beckwith & Choueiry 01]

- As a result, at each step during search, we have new larger partitions (bundles) with no more effort than necessary for FC

Dynamic bundling, too expensive to be practical?

Evaluation criteria

– Nodes Visited `NV`

– Constraints Checked `CC`

– Size of Solution Bundles `SB`

V_1 (d)

V_2 (e, f)

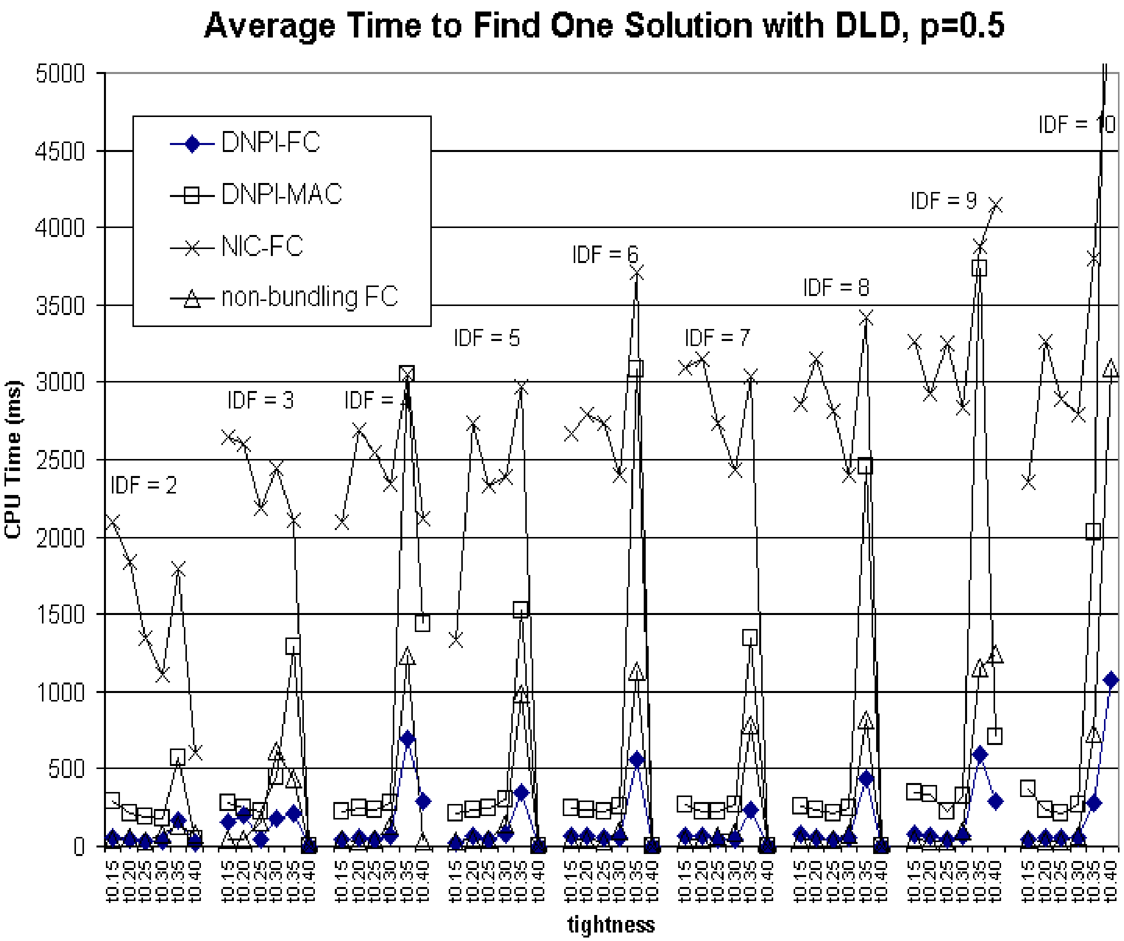
V_3 (a)

V_4 (b, c)

- Theoretical guarantees, *all solutions, static orderings*
  - – No more expensive than forward checking (`NV`, `CC`)
  - – No more expensive than static bundling (`NV`)
  - – Produces larger bundles than FC and static bundling (`SB`)
- Empirical tests, also measured CPU Time.

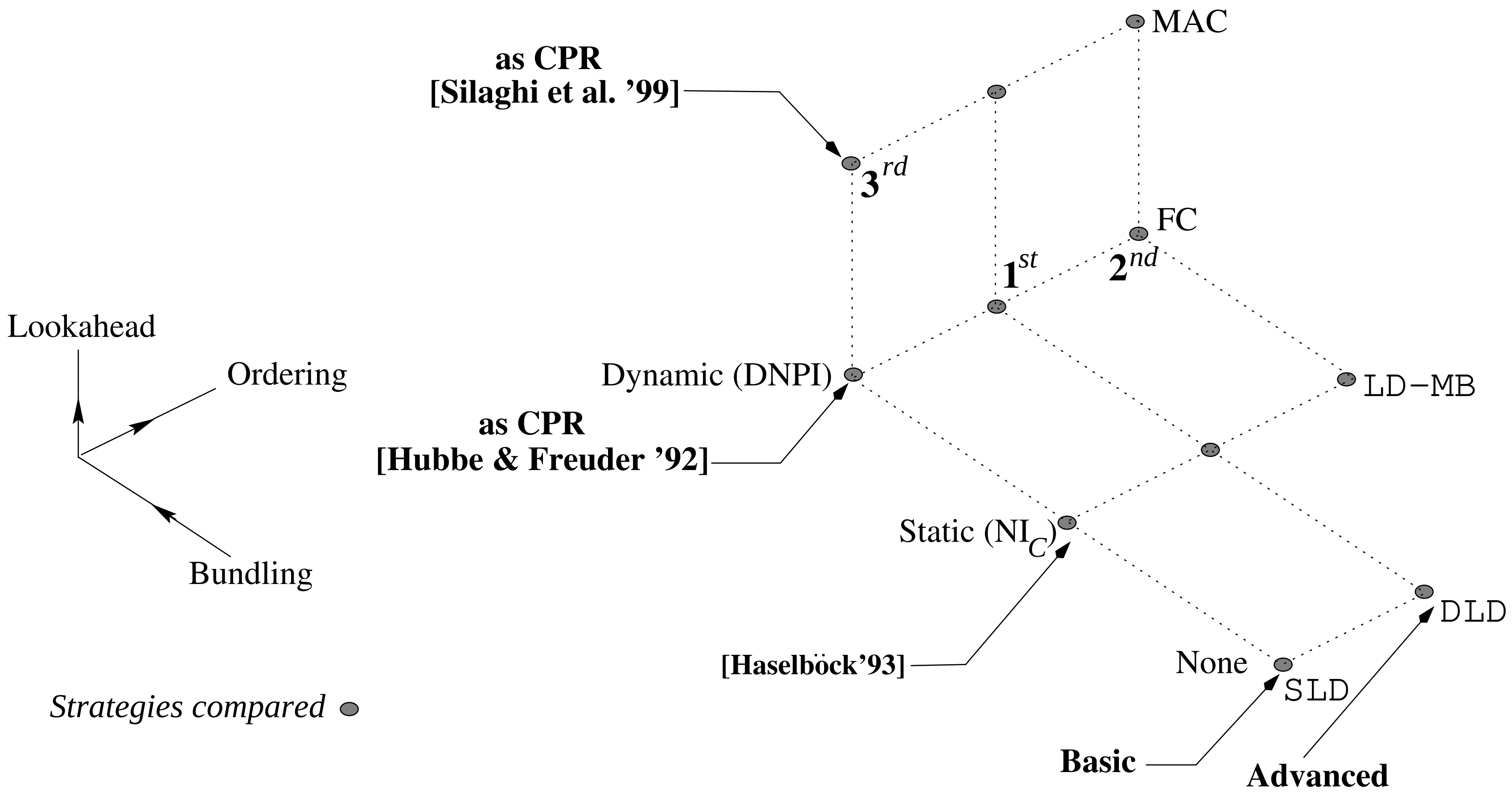
Results hold, even when no bundling is possible (e.g., puzzles)

Evidence: time (varying: tightness, IDF)



Novelty: reduce phase transition while . . .

Ranking of strategies



⟶ Extension to non-binary constraints [Lal & Choueiry 2003]

Summary

- Formalizes interchangeability in CSPs
- Introduces many types of interchangeability relations and show some relations among them
- Provides an algorithm for computing NI and k-interchangeability
- Other types of interchangeability:
  - NI_C: Haselboeck, IJCAI'93
  - CDI: Weigel, Faltings, Choueiry ECAI'96
  - NPI: Choueiry, Noubir, AAAI'98
  - DNPI: Choueiry, Beckwith 2001
- On-going: DNPI + search (dynamic bundling)