Title: Inference in First-Order Logic (II)
AIMA: Chapter 9 (Sections 9.5 to 9.6 pp.283)

Introduction to Artificial Intelligence
CSCE 476-876, Spring 2002
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Glossary 12:

Assigned: Friday, April 26, 2002
Due date: Monday , April 29, 2002
Note: Glossaries are optional.

- Completeness Theorem
- Generalized Resolution (disjunctions, implications)
- Implicative Normal Form
- Refutation
- Resolution Algorithm
- Semidecidable
- Skolemization
- Skolem Function

Outline:

- Completeness of GMP
- Generalization(s) of the resolution inference rule
- Resolution proof procedure
- Resolution + refutation proof procedure
- Conversion to normal form

Completeness of GMP:

Consider KB:

$$\forall x\ P(x) \Rightarrow Q(x)$$
$$\forall x\ \neg P(x) \Rightarrow R(x)$$
$$\forall x\ Q(x) \Rightarrow S(x)$$
$$\forall x\ R(x) \Rightarrow S(x)$$

Can we prove $S(A)$? GMP fails but..

Either, we have $P(A) \Rightarrow Q(A) \Rightarrow S(A)$
or, we have $\neg P(A) \Rightarrow R(A) \Rightarrow S(A)$

So KB $\models S(A)$, but GMP cannot allow derive it
$\longrightarrow$ GMP is an incomplete procedure

Reason: 2nd sentence is not a Horn clause

Task : find a complete proof procedure for FOL

Existence : established by Gödel in 1930/1931

There exists a set of inference rules that allows a complete proof procedure R

if $KB \models \alpha$ then $KB \vdash_R \alpha$

Resolution algorithm : Robinson, 1965

Modus Ponens: does not allow us to derive new implications

$$\frac{\alpha, \quad \alpha \Rightarrow \beta}{\beta}$$

Resolution rule: two forms, implications or disjunctions

$$\frac{\neg\alpha \Rightarrow \beta, \quad \beta \Rightarrow \gamma}{\neg\alpha \Rightarrow \gamma} \quad \text{or equivalently} \quad \frac{\alpha \vee \beta, \quad \neg\beta \vee \gamma}{\alpha \vee \gamma}$$

is more powerful than MP: allows us to derive a new implication by transitivity

Note: Resolution is a generalization of Modus Ponens:

$$\frac{\alpha, \quad \alpha \Rightarrow \beta}{\beta} \quad \text{is equivalent to} \quad \frac{True \Rightarrow \alpha, \quad \alpha \Rightarrow \beta}{True \Rightarrow \beta}$$

Resolution rule

$$\frac{\neg\alpha \Rightarrow \beta, \quad \beta \Rightarrow \gamma}{\neg\alpha \Rightarrow \gamma} \quad \text{or equivalently} \quad \frac{\alpha \vee \beta, \quad \neg\beta \vee \gamma}{\alpha \vee \gamma}$$

but also says:
either β is true and then γ necessarily true
or β is false then necessarily α is true
So, necessarily $\alpha \vee \gamma$ is true

Note: Premises have exactly two disjuncts
$\longrightarrow$ generalized versions: disjuncts of any length

Generalized resolution (disjunctions)

Given:
Premise 1: $p_1 \vee \ldots \vee p_j \vee \ldots \vee p_m$
Premise 2: $q_1 \vee \ldots \vee q_k \vee \ldots \vee q_n$

If $p_j = \neg q_k$, then we can derive the consequence:

$$p_1 \vee \ldots p_{j-1} \vee p_{j+1} \ldots \vee p_m \vee q_1 \vee q_{k-1} \vee q_{k+1} \ldots \vee q_n$$

More generally:
If Unify$(p_i, \neg q_k) = \theta$, from Premises 1, 2, we can derive:
$Subst(\theta, (p_1 \vee \ldots p_{j-1} \vee p_{j+1} \ldots \vee p_m \vee q_1 \ldots q_{k-1} \vee q_{k+1} \ldots \vee q_n))$.

Thus, the generalized resolution (disjunctions):

$$\frac{p_1 \vee \ldots p_j \ldots \vee p_m \quad q_1 \vee \ldots q_k \ldots \vee q_n}{Subst(\theta, (p_1 \vee \ldots p_{j-1} \vee p_{j+1} \ldots \vee p_m \vee q_1 \ldots q_{k-1} \vee q_{k+1} \ldots \vee q_n))}$$

Generalized resolution (disjunctions)

$$\frac{p_1 \vee \ldots p_j \ldots \vee p_m \quad q_1 \vee \ldots q_k \ldots \vee p_n}{Subst(\theta, (p_1 \vee \ldots p_{j-1} \vee p_{j+1} \ldots \vee p_m \vee q_1 \ldots q_{k-1} \vee q_{k+1} \ldots \vee q_n))}$$

Similarly, **Generalized resolution (implications)**

$$\frac{p_1 \wedge \ldots p_j \ldots \wedge p_{n_1} \Rightarrow r_1 \vee \ldots r_{n_2} \qquad s_1 \wedge \ldots \wedge s_{n_3} \Rightarrow q_1 \vee \ldots q_k \ldots \vee q_{n_4}}{Subst(\theta, (p_1 \wedge \ldots p_{j-1} \wedge p_{j+1} \wedge p_{n_1} \wedge s_1 \wedge \ldots s_{n_3} \Rightarrow r_1 \vee \ldots r_{n_2} \vee q_1 \vee \ldots q_{k-1} \vee q_{k+1} \vee \ldots \vee q_{n_4}))}$$

Conversion to conjunctive/implicative normal forms

Conjunctive normal form:
KB is a conjunction of disjunctions

'Implicative' normal form:
KB is a conjunction of implications

Consider KB:

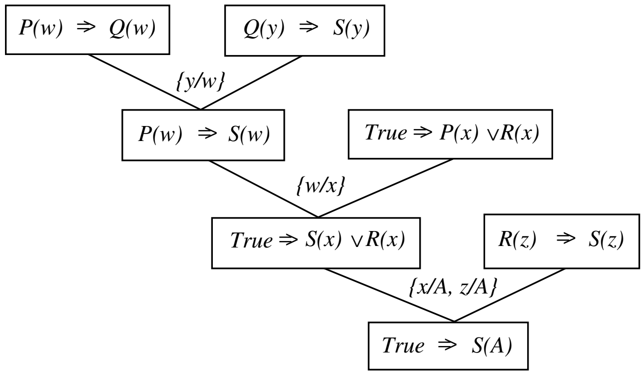
$$\forall x\ P(x) \Rightarrow Q(x)$$
$$\forall x\ \neg P(x) \Rightarrow R(x)$$
$$\forall x\ Q(x) \Rightarrow S(x)$$
$$\forall x\ R(x) \Rightarrow S(x)$$

Conjunctive Normal Form	Implicative Normal Form
historically more common	commonly used in AI
$\neg P(w) \vee Q(w)$	$P(w) \Rightarrow Q(w)$
$P(x) \vee R(x)$	$True \Rightarrow P(x) \vee R(x)$
$\neg Q(y) \vee S(y)$	$Q(y) \Rightarrow S(y)$
$\neg R(z) \vee S(z)$	$R(z) \Rightarrow S(z)$

Resolution proofs

$$P(w) \Rightarrow Q(w)$$
$$True \Rightarrow P(x) \vee R(x)$$
$$Q(y) \Rightarrow S(y)$$
$$R(z) \Rightarrow S(z)$$

Prove $S(A)$ using resolution inference rule and forward chaining (backward chaining is OK too)



Technically, $True \Rightarrow S(A) \vee S(A)$
Use factoring rule: $True \Rightarrow S(A)$

Question

Is resolution is a complete proof procedure?

Consider empty KB, try proving: $P \vee \neg P$ (valid sentence)

Since KB is empty, no way to use KB to prove the sentence

$\longrightarrow$ use resolution with <u>refutation</u>
$\longrightarrow$ resolution with refutation is a complete proof procedure

Refutation

a.k.a. proof by contradiction, reductio ad absurdum

Idea: want to proof P ? assume P false, and prove contradiction

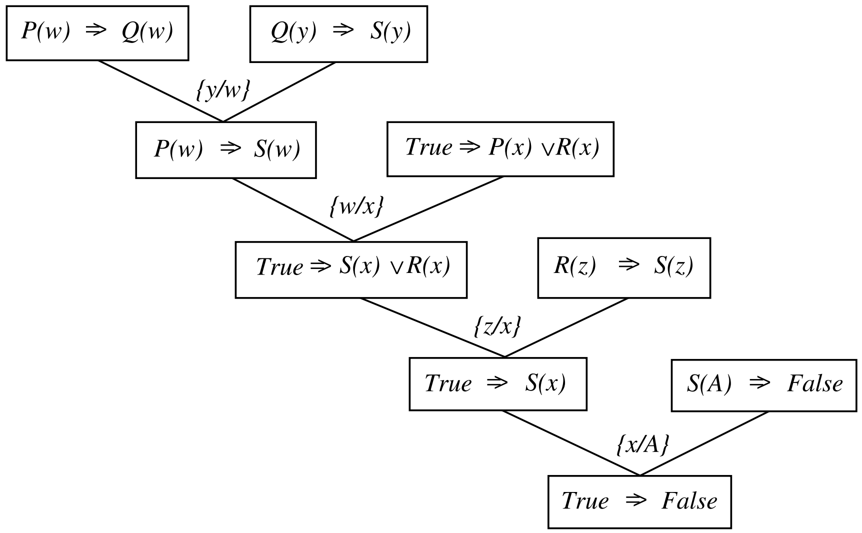
$$(KB \wedge \neg P \Rightarrow False) \Leftrightarrow (KB \Rightarrow P)$$

Note: $\neg P$ in 'implicative' normal form is written: $P \Rightarrow False$

Resolution + refutation: Example

KB (Implicative Normal Form):

$$P(w) \Rightarrow Q(w)$$
$$True \Rightarrow P(x) \vee R(x)$$
$$Q(y) \Rightarrow S(y)$$
$$R(z) \Rightarrow S(z)$$
$$S(A) \Rightarrow False$$



Conversion to Normal Form

7/8 rules to execute in sequence (easy to understand, hard to remember):

1. Eliminate implications
2. Move $\neg$ inwards
3. Standardize variables
4. Move quantifiers left
5. Skolemize
6. Distribute $\wedge$ over $\vee$
7. Flatten nested conjunctions and disjunctions
8. Convert disjunctions to implications

1. **Eliminate implications**:

$$p \Rightarrow q \quad \text{becomes} \quad \neg p \vee q$$

2. **Move $\neg$ inwards** :
   negation is allowed only on atoms, use Morgan's laws

$\neg(p \vee q)$	becomes	$\neg p \wedge \neg q$
$\neg(p \wedge q)$	becomes	$\neg p \vee \neg q$
$\neg \forall x, p$	becomes	$\exists x, \neg p$
$\neg \exists x, p$	becomes	$\forall x, \neg p$
$\neg\neg p$	becomes	p

3. **Standardize variables**:
   (to avoid confusion when dropping quantifiers)
   Example: $(\forall x P(x)) \vee (\exists x Q(x))$ becomes $(\forall x P(x)) \vee (\exists y Q(y))$

4. **Move quantifiers left**:
   Keep the order in which they appear!
   Example: $(p \vee \forall\, x\; q)$ becomes $\forall\, x\, p \vee q$

this is true because p is guaranteed not to contain X (step 3)

4. **Skolemization**:

this is the process of moving existential quantifiers by elimination

Simple case, as in existential elimination (remember?)
$\exists\, x P(x)$ translates into $P(A)$, $\quad A$ constant, not appearing in KB

More complex case: Everyone has a heart
$\forall x\; Person(x) \Rightarrow \exists\, y\; Heart(y) \wedge Has(x, y)$
Apply 'simple' existential elimination, replace y with a constant H
$\forall x\; Person(x) \Rightarrow Heart(H) \wedge Has(x, H)$
which means: everyone has the same heart H :—(

Skolem function: $F(x)$ heart of x, F is a function whose name does not appear elsewhere in the KB
$\forall x\; Person(x) \Rightarrow Heart(F(x)) \wedge Has(x, F(x))$

Original sentence: $\forall x \; Person(x) \Rightarrow \exists \, y \; Heart(y) \wedge Has(x,y)$

Skolemized sentence:
$\forall x \; Person(x) \Rightarrow Heart(F(x)) \wedge Has(x, F(x))$

Skolemization: replace the <u>existentially</u> quantified variable by a Skolem function applied to <u>all</u> the variables <u>universally</u> quantified outside the existentially quantified variable.

$\longrightarrow$ After Skolemization, existentially quantified variables are eliminated, so we can drop universal quantifiers as all variables are now only universally quantified..

6. **Distribute $\wedge$ over $\vee$** :
   $(a \wedge b) \vee c$ becomes $(a \vee c) \wedge (b \vee c)$

7. **Flatten nested conjunctions and disjunctions**:
   $(a \wedge b) \wedge c$ becomes $(a \wedge b \wedge c)$
   $(a \vee b) \vee c$ becomes $(a \vee b \vee c)$

8. **Convert disjunctions to implications**: (optional)
   $(\neg a \vee \neg b \vee c \vee d)$ becomes $a \wedge b \;\Rightarrow\; c \vee d$

Summary

Propositional logic:

7 Inference rules: complete and sound
general forms: NP-Complete
Horn clauses: tractable

FOL: (predicate logic)
Generalized Modus Ponens (And-intro+Universal elimination+MP):
— Requires KB to be in Horn Normal Form, otherwise incomplete
— forward+backward chaining

Resolution + Refutation:
Complete proof procedure
Sentences need to be in Normal Form (conjunctive or implicative) using a 7/8-step recipe