Predicate Logic and Quantifiers

Predicate Logic and Quantifiers

CSE235

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Computer Science & Engineering 235
Introduction to Discrete Mathematics
Sections 1.3–1.4 of Rosen

Notes

Introduction

Predicate Logic and Quantifiers

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Consider the following statements:

$$x > 3, \quad x = y + 3, \quad x + y = z$$

The truth value of these statements has no meaning without specifying the values of x, y, z.

However, we *can* make propositions out of such statements.

A *predicate* is a property that is affirmed or denied about the *subject* (in logic, we say "variable" or "argument") of a *statement*.

$$\text{“}\underbrace{x}_{\text{subject}}\ \underbrace{\text{is greater than 3”}}_{\text{predicate}}$$

Terminology: affirmed = holds = is true; denied = does not hold = is not true.

Notes

Propositional Functions

Predicate Logic and Quantifiers

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To write in predicate logic:

$$\text{“}\underbrace{x}_{\text{subject}}\ \underbrace{\text{is greater than 3”}}_{\text{predicate}}$$

We introduce a (functional) symbol for the predicate, and put the subject as an argument (to the functional symbol): $P(x)$

Examples:

- Father(x): unary predicate
- Brother(x,y): binary predicate
- Sum(x,y,z): ternary predicate
- P(x,y,z,t): n-ary predicate

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Propositional Functions

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Definition

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the *propositional function* P. Here, $(x_1, x_2, \ldots, x_n)$ is an n-tuple and P is a predicate.

You can think of a propositional function as a function that

- Evaluates to true or false.
- Takes one or more arguments.
- Expresses a predicate involving the argument(s).
- Becomes a proposition when values are assigned to the arguments.

Notes

Propositional Functions
Example

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Example

Let $Q(x, y, z)$ denote the statement "$x^2 + y^2 = z^2$". What is the truth value of $Q(3, 4, 5)$? What is the truth value of $Q(2, 2, 3)$? How many values of (x, y, z) make the predicate true?

Notes

Propositional Functions
Example

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Example

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Since $3^2 + 4^2 = 25 = 5^2$, $Q(3, 4, 5)$ is true.

Since $2^2 + 2^2 = 8 \neq 3^2 = 9$, $Q(2, 2, 3)$ is false.

There are infinitely many values for (x, y, z) that make this propositional function true—how many right triangles are there?

Notes

Universe of Discourse

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Consider the previous example. Does it make sense to assign to x the value "blue"?

Intuitively, the *universe of discourse* is the set of all things we wish to talk about; that is, the set of all objects that we can sensibly assign to a variable in a propositional function.

What would be the universe of discourse for the propositional function $P(x) =$ "The test will be on x the 23rd" be?

Notes

Universe of Discourse
Multivariate Functions

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Moreover, each variable in an n-tuple may have a different universe of discourse.

Let $P(r, g, b, c) =$ "The rgb-value of the color c is (r, g, b)".

For example, $P(255, 0, 0, red)$ is true, while $P(0, 0, 255, green)$ is false.

What are the universes of discourse for (r, g, b, c)?

Notes

Quantifiers
Introduction

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A predicate becomes a proposition when we assign it fixed values. However, another way to make a predicate into a proposition is to *quantify* it. That is, the predicate is true (or false) for *all* possible values in the universe of discourse or for *some* value(s) in the universe of discourse.

Such *quantification* can be done with two *quantifiers*: the *universal* quantifier and the *existential* quantifier.

Notes

Universal Quantifier
Definition

Definition

The *universal quantification* of a predicate $P(x)$ is the proposition "$P(x)$ is true for all values of x in the universe of discourse" We use the notation

$$\forall x P(x)$$

which can be read "for all x"

If the universe of discourse is finite, say $\{n_1, n_2, \ldots, n_k\}$, then the universal quantifier is simply the conjunction of all elements:

$$\forall x P(x) \iff P(n_1) \wedge P(n_2) \wedge \cdots \wedge P(n_k)$$

Notes

Universal Quantifier
Example I

- Let $P(x)$ be the predicate "x must take a discrete mathematics course" and let $Q(x)$ be the predicate "x is a computer science student".
- The universe of discourse for both $P(x)$ and $Q(x)$ is all UNL students.
- Express the statement "Every computer science student must take a discrete mathematics course".
- Express the statement "Everybody must take a discrete mathematics course or be a computer science student".

Notes

Universal Quantifier
Example I

- Let $P(x)$ be the predicate "x must take a discrete mathematics course" and let $Q(x)$ be the predicate "x is a computer science student".
- The universe of discourse for both $P(x)$ and $Q(x)$ is all UNL students.
- Express the statement "Every computer science student must take a discrete mathematics course".

$$\forall x (Q(x) \rightarrow P(x))$$

- Express the statement "Everybody must take a discrete mathematics course or be a computer science student".

Notes

Universal Quantifier
Example I

Predicate Logic and Quantifiers

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$$\forall x(Q(x) \vee P(x))$$

Notes

Universal Quantifier
Example I

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- Express the statement "Every computer science student must take a discrete mathematics course".

$$\forall x(Q(x) \rightarrow P(x))$$

- Express the statement "Everybody must take a discrete mathematics course or be a computer science student".

$$\forall x(Q(x) \vee P(x))$$

- Are these statements true or false?

Notes

Universal Quantifier
Example II

Predicate Logic and Quantifiers

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Express the statement "for every x and for every y, $x + y > 10$"

Notes

Universal Quantifier
Example II

Predicate Logic and Quantifiers

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Express the statement "for every x and for every y, $x + y > 10$"

Let $P(x, y)$ be the statement $x + y > 10$ where the universe of discourse for x, y is the set of integers.

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Universal Quantifier
Example II

Predicate Logic and Quantifiers

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Express the statement "for every x and for every y, $x + y > 10$"

Let $P(x, y)$ be the statement $x + y > 10$ where the universe of discourse for x, y is the set of integers.

Answer:

$$\forall x \forall y P(x, y)$$

Notes

Universal Quantifier
Example II

Predicate Logic and Quantifiers

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Express the statement "for every x and for every y, $x + y > 10$"

Let $P(x, y)$ be the statement $x + y > 10$ where the universe of discourse for x, y is the set of integers.

Answer:

$$\forall x \forall y P(x, y)$$

Note that we can also use the shorthand

$$\forall x, y P(x, y)$$

Notes

Existential Quantifier
Definition

Definition

The *existential quantification* of a predicate $P(x)$ is the proposition "There exists an x in the universe of discourse such that $P(x)$ is true." We use the notation

$$\exists x P(x)$$

which can be read "there exists an x"

Again, if the universe of discourse is finite, $\{n_1, n_2, \ldots, n_k\}$, then the existential quantifier is simply the disjunction of all elements:

$$\exists x P(x) \iff P(n_1) \vee P(n_2) \vee \cdots \vee P(n_k)$$

Notes

Existential Quantifier
Example I

Let $P(x, y)$ denote the statement, "$x + y = 5$".

What does the expression,

$$\exists x \exists y P(x)$$

mean?

What universe(s) of discourse make it true?

Notes

Existential Quantifier
Example II

Express the statement "there exists a real solution to $ax^2 + bx - c = 0$"

Notes

Existential Quantifier
Example II

Predicate Logic and Quantifiers

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Express the statement "there exists a real solution to $ax^2 + bx - c = 0$"

Let $P(x)$ be the statement $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ where the universe of discourse for x is the set of reals. Note here that a, b, c are all fixed constants.

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Existential Quantifier
Example II

Predicate Logic and Quantifiers

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Express the statement "there exists a real solution to $ax^2 + bx - c = 0$"

Let $P(x)$ be the statement $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ where the universe of discourse for x is the set of reals. Note here that a, b, c are all fixed constants.

The statement can thus be expressed as

$$\exists x P(x)$$

Notes

Existential Quantifier
Example II Continued

Predicate Logic and Quantifiers

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Question: what is the truth value of $\exists x P(x)$?

Notes

Existential Quantifier
Example II Continued

Predicate Logic and Quantifiers

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Question: what is the truth value of $\exists x P(x)$?

Answer: it is false. For any real numbers such that $b^2 < 4ac$, there will only be complex solutions, for these cases no such *real* number x can satisfy the predicate.

How can we make it so that it *is* true?

Notes

Existential Quantifier
Example II Continued

Predicate Logic and Quantifiers

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Question: what is the truth value of $\exists x P(x)$?

Answer: it is false. For any real numbers such that $b^2 < 4ac$, there will only be complex solutions, for these cases no such *real* number x can satisfy the predicate.

How can we make it so that it *is* true?

Answer: change the universe of discourse to the complex numbers, $\mathbb{C}$.

Notes

Quantifiers
Truth Values

Predicate Logic and Quantifiers

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In general, when are quantified statements true/false?

Statement	True When	False When
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Table: Truth Values of Quantifiers

Notes

Mixing Quantifiers I

Existential and universal quantifiers can be used together to quantify a predicate statement; for example,

$$\forall x \exists y P(x, y)$$

is perfectly valid. However, you must be careful—it must be read left to right.

For example, $\forall x \exists y P(x, y)$ is not equivalent to $\exists y \forall x P(x, y)$. Thus, ordering is important.

Notes

Mixing Quantifiers II

For example:

- $\forall x \exists y Loves(x, y)$: everybody loves somebody
- $\exists y \forall x Loves(x, y)$: There is someone loved by everyone

Those expressions do not mean the same thing!

Note that $\exists y \forall x P(x, y) \rightarrow \forall x \exists y P(x, y)$, but the converse does not hold

However, you *can* commute *similar* quantifiers; $\exists x \exists y P(x, y)$ is equivalent to $\exists y \exists x P(x, y)$ (which is why our shorthand was valid).

Notes

Mixing Quantifiers
Truth Values

Statement	True When	False When
$\forall x \forall y P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is at least one pair, x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x, there is a y for which $P(x, y)$ is true.	There is an x for which $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x, there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$	There is at least one pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Table: Truth Values of 2-variate Quantifiers

Notes

Mixing Quantifiers
Example I

Predicate Logic and Quantifiers

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Express, in predicate logic, the statement that there are an infinite number of integers.

Notes

Mixing Quantifiers
Example I

Predicate Logic and Quantifiers

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Express, in predicate logic, the statement that there are an infinite number of integers.

Let $P(x, y)$ be the statement that $x < y$. Let the universe of discourse be the integers, $\mathbb{Z}$.

Notes

Mixing Quantifiers
Example I

Predicate Logic and Quantifiers

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Express, in predicate logic, the statement that there are an infinite number of integers.

Let $P(x, y)$ be the statement that $x < y$. Let the universe of discourse be the integers, $\mathbb{Z}$.

Then the statement can be expressed by the following.

$$\forall x \exists y P(x, y)$$

Notes

Mixing Quantifiers
Example II: More Mathematical Statements

Predicate Logic and Quantifiers

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Express the *commutative law of addition* for $\mathbb{R}$.

Notes

Mixing Quantifiers
Example II: More Mathematical Statements

Predicate Logic and Quantifiers

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Express the *commutative law of addition* for $\mathbb{R}$.

We want to express that for every pair of reals, x, y the following identity holds:

$$x + y = y + x$$

Notes

Mixing Quantifiers
Example II: More Mathematical Statements

Predicate Logic and Quantifiers

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Express the *commutative law of addition* for $\mathbb{R}$.

We want to express that for every pair of reals, x, y the following identity holds:

$$x + y = y + x$$

Then we have the following:

$$\forall x \forall y (x + y = y + x)$$

Notes

Mixing Quantifiers
Example II: More Mathematical Statements Continued

Predicate Logic and Quantifiers

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Express the *multiplicative inverse law* for (nonzero) rationals $\mathbb{Q} \setminus \{0\}$.

Notes

Mixing Quantifiers
Example II: More Mathematical Statements Continued

Predicate Logic and Quantifiers

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Express the *multiplicative inverse law* for (nonzero) rationals $\mathbb{Q} \setminus \{0\}$.

We want to express that for every real number x, there exists a real number y such that $xy = 1$.

Notes

Mixing Quantifiers
Example II: More Mathematical Statements Continued

Predicate Logic and Quantifiers

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Express the *multiplicative inverse law* for (nonzero) rationals $\mathbb{Q} \setminus \{0\}$.

We want to express that for every real number x, there exists a real number y such that $xy = 1$.

Then we have the following:

$$\forall x \exists y (xy = 1)$$

Notes

Mixing Quantifiers

Example II: False Mathematical Statements

Is commutativity for subtraction valid over the reals?

Notes

Mixing Quantifiers

Example II: False Mathematical Statements

Is commutativity for subtraction valid over the reals?

That is, for all pairs of real numbers x, y does the identity $x - y = y - x$ hold? Express this using quantifiers.

Notes

Mixing Quantifiers

Example II: False Mathematical Statements

Is commutativity for subtraction valid over the reals?

That is, for all pairs of real numbers x, y does the identity $x - y = y - x$ hold? Express this using quantifiers.

The expression is

$$\forall x \forall y (x - y = y - x)$$

Notes

Mixing Quantifiers

Example II: False Mathematical Statements Continued

Predicate Logic and Quantifiers

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Is there a multiplicative inverse law over the nonzero integers?

Notes

Mixing Quantifiers

Example II: False Mathematical Statements Continued

Predicate Logic and Quantifiers

CSE235

Is there a multiplicative inverse law over the nonzero integers?

That is, for every integer x does there exists a y such that $xy = 1$?

Notes

Mixing Quantifiers

Example II: False Mathematical Statements Continued

Predicate Logic and Quantifiers

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Is there a multiplicative inverse law over the nonzero integers?

That is, for every integer x does there exists a y such that $xy = 1$?

This is false, since we can find a *counter example*. Take any integer, say 5 and multiply it with another integer, y. If the statement held, then $5 = 1/y$, but for any (nonzero) integer y, $|1/y| \leq 1$.

Notes

Mixing Quantifiers
Exercise

Predicate Logic and Quantifiers

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Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

Notes

Mixing Quantifiers
Exercise

Predicate Logic and Quantifiers

CSE235

Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

- Let $P(x, y)$ be the expression "$x + y = y$".

Notes

Mixing Quantifiers
Exercise

Predicate Logic and Quantifiers

CSE235

Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

- Let $P(x, y)$ be the expression "$x + y = y$".
- Let $Q(x, y)$ be the expression "$xy = x$".

Notes

Mixing Quantifiers
Exercise

Predicate Logic and Quantifiers

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Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

- Let $P(x, y)$ be the expression "$x + y = y$".
- Let $Q(x, y)$ be the expression "$xy = x$".
- Then the expression is

$$\exists x \forall y \, (P(x, y) \wedge Q(x, y))$$

Notes

Mixing Quantifiers
Exercise

Predicate Logic and Quantifiers

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Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

- Let $P(x, y)$ be the expression "$x + y = y$".
- Let $Q(x, y)$ be the expression "$xy = x$".
- Then the expression is

$$\exists x \forall y \, (P(x, y) \wedge Q(x, y))$$

- Over what universe(s) of discourse does this statement hold?

Notes

Mixing Quantifiers
Exercise

Predicate Logic and Quantifiers

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Express the statement "there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x" as a logical expression.

Solution:

- Let $P(x, y)$ be the expression "$x + y = y$".
- Let $Q(x, y)$ be the expression "$xy = x$".
- Then the expression is

$$\exists x \forall y \, (P(x, y) \wedge Q(x, y))$$

- Over what universe(s) of discourse does this statement hold?
- This is the *additive identity law* and holds for $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{Q}$ but does not hold for $\mathbb{Z}^+$.

Notes

Binding Variables I

Predicate Logic and Quantifiers

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When a quantifier is used on a variable x, we say that x is *bound*. If no quantifier is used on a variable in a predicate statement, it is called *free*.

Example

In the expression $\exists x \forall y P(x, y)$ both x and y are bound.
In the expression $\forall x P(x, y)$, x is bound, but y is free.

A statement is called a *well-formed formula*, when all variables are properly quantified.

Notes

Binding Variables II

Predicate Logic and Quantifiers

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The set of all variables bound by a common quantifier is the *scope* of that quantifier.

Example

In the expression $\exists x, y \forall z P(x, y, z, c)$ the scope of the existential quantifier is $\{x, y\}$, the scope of the universal quantifier is just z and c has no scope since it is free.

Notes

Negation

Predicate Logic and Quantifiers

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Just as we can use negation with propositions, we can use them with quantified expressions.

Lemma

Let $P(x)$ be a predicate. Then the following hold.

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

This is essentially a quantified version of De Morgan's Law (in fact if the universe of discourse is finite, it is *exactly* De Morgan's law).

Notes

Negation
Truth Values

Predicate Logic and Quantifiers

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Statement	True When	False When
$\neg\exists x P(x) \equiv \forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x) \equiv \exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Table: Truth Values of Negated Quantifiers

Notes

Prolog

Predicate Logic and Quantifiers

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Prolog (Programming in Logic) is a programming language based on (a restricted form of) Predicate Calculus. It was developped by the logicians of the artificial intelligence community for symbolic reasoning.

- Prolog allows the user to express facts and rules
- Facts are propositional functions: student(juana), enrolled(juana,cse235), instructor(patel,cse235), etc.
- Rules are implications with conjunctions: teaches(X,Y) :- instructor(X,Z), enrolled(Y,Z)
- Prolog answers queries such as:
  ?enrolled(juana,cse478)
  ?enrolled(X,cse478)
  ?teaches(X,juana)
  by binding variables and doing theorem proving (i.e., applying inference rules) as we will see in Section 1.5.

Notes

English into Logic

Predicate Logic and Quantifiers

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Logic is more precise than English.

Transcribing English to Logic and vice versa can be tricky.

When writing statements with quantifiers, *usually* the correct meaning is conveyed with the following combinations:

- Use $\forall$ with $\Rightarrow$
  Example: $\forall x Lion(x) \Rightarrow Fierce(x)$
  $\forall x Lion(x) \wedge Fierce(x)$ means "everyone is a lion and everyone is fierce"
- Use $\exists$ with $\wedge$
  Example: $\exists x Lion(x) \wedge Drinks(x, coffee)$: holds when you have at least one lion that drinks coffee
  $\exists x Lion(x) \Rightarrow Drinks(x, coffee)$ holds when you have people even though no lion drinks coffee.

Notes

Conclusion

Predicate Logic and Quantifiers

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Examples? Exercises?

- Rewrite the expression, $\neg\forall x\big(\exists y\forall z P(x,y,z) \land \exists z\forall y P(x,y,z)\big)$
- Let $P(x,y)$ denote "x is a factor of y" where $x \in \{1,2,3,\ldots\}$ and $y \in \{2,3,4,\ldots\}$. Let $Q(y)$ denote "$\forall x\big[P(x,y) \to ((x=y) \lor (x=1))\big]$". When is $Q(y)$ true?

Notes

Conclusion

Predicate Logic and Quantifiers

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Examples? Exercises?

- Rewrite the expression, $\neg\forall x\big(\exists y\forall z P(x,y,z) \land \exists z\forall y P(x,y,z)\big)$
- Answer: Use the negated quantifiers and De Morgan's law.

$$\exists x\big(\forall y\exists z\neg P(x,y,z) \lor \forall z\exists y\neg P(x,y,z)\big)$$

- Let $P(x,y)$ denote "x is a factor of y" where $x \in \{1,2,3,\ldots\}$ and $y \in \{2,3,4,\ldots\}$. Let $Q(y)$ denote "$\forall x\big[P(x,y) \to ((x=y) \lor (x=1))\big]$". When is $Q(y)$ true?

Notes

Conclusion

Predicate Logic and Quantifiers

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Examples? Exercises?

- Rewrite the expression, $\neg\forall x\big(\exists y\forall z P(x,y,z) \land \exists z\forall y P(x,y,z)\big)$
- Answer: Use the negated quantifiers and De Morgan's law.

$$\exists x\big(\forall y\exists z\neg P(x,y,z) \lor \forall z\exists y\neg P(x,y,z)\big)$$

- Let $P(x,y)$ denote "x is a factor of y" where $x \in \{1,2,3,\ldots\}$ and $y \in \{2,3,4,\ldots\}$. Let $Q(y)$ denote "$\forall x\big[P(x,y) \to ((x=y) \lor (x=1))\big]$". When is $Q(y)$ true?
- Answer: Only when y is a prime number.

Notes

Extra Question

Some students wondered if

$$\forall x, y P(x, y) \equiv \forall x P(x, y) \wedge \forall y P(x, y)$$

This is certainly not true. In the left-hand side, both x and y are bound. In the right-hand side, x is bound in the first predicate, but y is free. In the second predicate, y is bound but x is free.

All variables that occur in a propositional function must be bound to turn it into a proposition.

Thus, the left-hand side is a proposition, but the right-hand side is not. How can they be equivalent?

Notes