

# Introduction to NP-Complete Problems

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Definitions

The 4-Step Proof

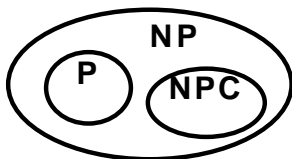
Example 1: Vertex Cover

Example 2: Jogger

# $\mathcal{P}$ , $\mathcal{NP}$ and $\mathcal{NP}$ -Complete

Given a problem, it belongs to  $\mathcal{P}$ ,  $\mathcal{NP}$  or  $\mathcal{NP}$ -Complete classes, if:

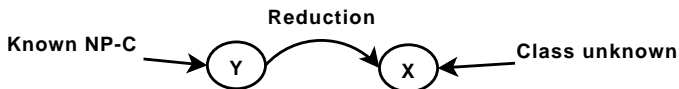
- ▶  $\mathcal{NP}$ : verifiable in polynomial time.
- ▶  $\mathcal{P}$ : decidable in polynomial time.
- ▶  $\mathcal{NP}$ -Complete: all problems in  $\mathcal{NP}$  can be reduced to it in polynomial time.



# The 4-Step Proof

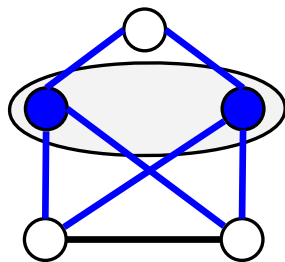
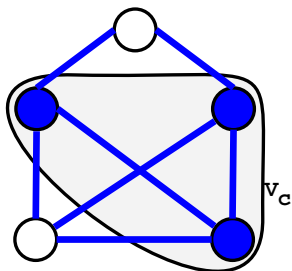
Given a problem  $X$ , prove it is in  $\mathcal{NP}$ -Complete.

1. Prove  $X$  is in  $\mathcal{NP}$ .
2. Select problem  $Y$  that is known to be in  $\mathcal{NP}$ -Complete.
3. Define a polynomial time reduction from  $Y$  to  $X$ .
4. Prove that given an instance of  $Y$ ,  $Y$  has a solution *iff*  $X$  has a solution.



# Vertex Cover

- ▶ A *vertex cover* of a graph  $G = (V, E)$  is a  $V_C \subseteq V$  such that every  $(a, b) \in E$  is incident to at least a  $u \in V_C$ .

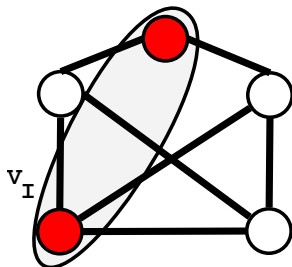


→ Vertices in  $V_C$  'cover' all the edges of  $G$ .

- ▶ The VERTEX COVER (VC) decision problem:  
Does  $G$  have a vertex cover of size  $k$ ?

# Independent Set

- ▶ An *independent set* of a graph  $G = (V, E)$  is a  $V_I \subseteq V$  such that no two vertices in  $V_I$  share an edge.



→  $u, v \in V_I$  cannot be neighbors.

- ▶ The INDEPENDENT SET (IS) decision problem:  
Does  $G$  have an independent set of size  $k$ ?

# Prove VERTEX COVER is $\mathcal{NP}$ -complete

Given that the INDEPENDENT SET (IS) decision problem is  $\mathcal{NP}$ -complete, prove that VERTEX COVER (VC) is  $\mathcal{NP}$ -complete.  
Solution:

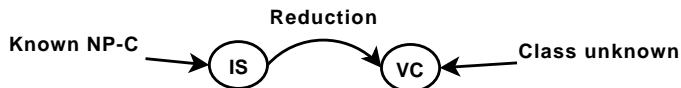
1. Prove VERTEX COVER is in  $\mathcal{NP}$ .

- ▶ Given  $V_C$ , vertex cover of  $G = (V, E)$ ,  $|V_C| = k$
- ▶ We can check in  $O(|E| + |V|)$  that  $V_C$  is a vertex cover for  $G$ .  
How?
  - ▶ For each vertex  $v \in V_C$ , remove all incident edges.
  - ▶ Check if all edges were removed from  $G$ .
- ▶ Thus, VERTEX COVER  $\in \mathcal{NP}$

## Prove VERTEX COVER is $\mathcal{NP}$ -complete (2)

2. Select a known  $\mathcal{NP}$ -complete problem.

- ▶ INDEPENDENT SET (IS) is a known  $\mathcal{NP}$ -complete problem.
- ▶ Use IS to prove that VC is  $\mathcal{NP}$ -complete.



## Prove VERTEX COVER is $\mathcal{NP}$ -complete (3)

3. Define a polynomial-time reduction from IS to VC:

- ▶ Given a general instance of IS:  $G'=(V', E')$ ,  $k'$
- ▶ Construct a specific instance of VC:  $G=(V, E)$ ,  $k$ 
  - ▶  $V=V'$
  - ▶  $E=E'$
  - ▶  $(G=G')$
  - ▶  $k=|V'| - k'$
- ▶ This transformation is polynomial:
  - ▶ Constant time to construct  $G=(V, E)$
  - ▶  $O(|V|)$  time to count the number of vertices
- ▶ Prove that there is a  $V_I$  ( $|V_I| = k'$ ) for  $G'$  iff there is an  $V_C$  ( $|V_C| = k$ ) for  $G$ .

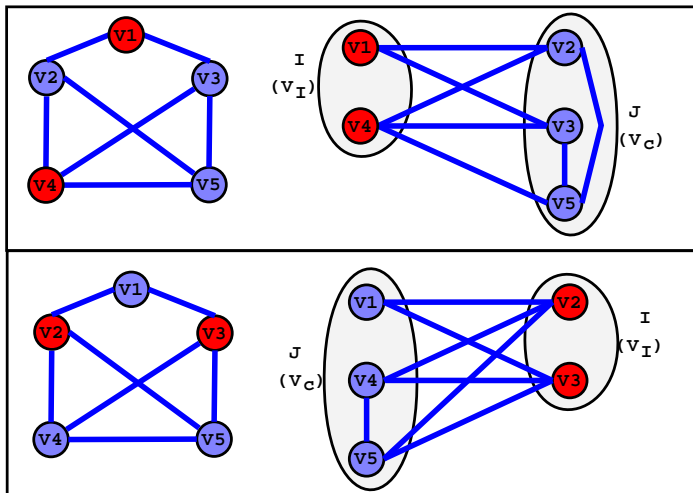
## Prove VERTEX COVER is $\mathcal{NP}$ -complete (4)

Prove  $G'$  has an independent set  $V_I$  of size  $k'$  iff VC has a vertex cover  $V_C$  of size  $k$ .

- ▶ Consider two sets  $I$  and  $J$  s.t.  $I \cap J = \emptyset$  and  $I \cup J = V = V'$
- ▶ Given any edge  $(u, v)$ , one of the following four cases holds:
  1.  $u, v \in I$
  2.  $u \in I$  and  $v \in J$
  3.  $u \in J$  and  $v \in I$
  4.  $u, v \in J$
- ▶ Assume that  $I$  is an independent set of  $G'$  then:
  - ▶ Case 1 cannot be; (vertices in  $I$  cannot be adjacent)
  - ▶ In cases 2 and 3,  $(u, v)$  has *exactly one* endpoint in  $J$ .
  - ▶ In case 4,  $(u, v)$  has *both* endpoints in  $J$ .
  - ▶ In cases 2, 3 and 4,  $(u, v)$  has *at least one* endpoint  $J$ .
  - ▶ Thus, vertices in  $J$  cover all edges of  $G'$ .
  - ▶ Also:  $|I| = |V| - |J|$  since  $I \cap J = \emptyset$  and  $I \cup J = V = V'$
  - ▶ Thus, if  $I$  is an independent set of  $G'$ , then  $J$  is a vertex cover of  $G' (= G)$ .

Similarly, we can prove that if  $J$  is a vertex cover for  $G'$ , then  $I$  is an independent set for  $G'$ . □

# Prove VERTEX COVER is $\mathcal{NP}$ -complete (5)



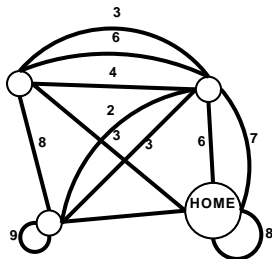
## Jogger Problem

Given a weighted, undirected graph  $G$  with:

- ▶ loops, multiple edges, and only positive weights,
- ▶ a special node  $v$  called *home*,
- ▶ and given an integer  $i \geq 0$ .

Is there a route for a jogger  $J$  that:

- ▶ starts from home,
- ▶ travels a distance  $i$ , and
- ▶ returns home
- ▶ without repeating an edge (nodes can be repeated)?



## Jogger is $\mathcal{NP}$ -complete

1. Jogger is in  $\mathcal{NP}$ : Given a path  $P$ , we can check in  $O(|P|)$  whether or not the sum of all edge weights is equal to  $i$ .
2. Consider the Subset Sum (SS) problem<sup>1</sup>, which is a known  $\mathcal{NP}$ -complete problem.

Given a set  $S$  of positive integers, is there a subset  $S' \subseteq S$  such that sum of the elements of  $S'$  is  $t$ .

Example:  $S = \{1, 3, 4, 5, 7, 8\}$ , find  $S' \subseteq S$  such that sum of the elements in  $S'$  is 15.

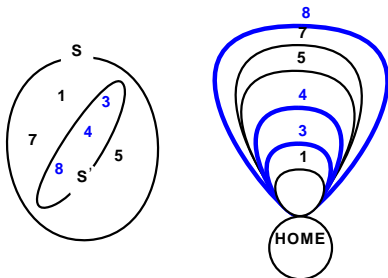
3. Reduce Subset Sum to the Jogger.

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<sup>1</sup>One type of knapsack problem.

## Jogger is $\mathcal{NP}$ -complete

- ▶ Given an instance of SS:  $S = \{a_1, a_2, \dots, a_n\}$ , construct a graph  $G$  as follows:
  - ▶  $G$  has a unique node,  $v$ , which is the home.
  - ▶ For each  $a_i \in S$ , add a self-loop to  $v$  of weight  $a_i$ .
  - ▶ Let  $i$  (of the Jogger) =  $t$  (of the Sum Set).
- ▶ This construction is obviously linear in the number of elements in  $S$ .



## Jogger is $\mathcal{NP}$ -complete (2)

4. SS has a solution *iff* Jogger has a solution.
- ▶  $G$  contains a path starting from home, never repeating an edge, and returning back home with a total distance exactly  $i$  *iff*  $S$  has a subset  $S'$  with sum of elements of  $S'$  equal to  $t$ .
  - ▶ If  $S' \subseteq S$  is a solution to SS, then the Jogger has a path of length  $i = t$  by taking the edges (loops) corresponding to the elements in  $S'$ .
  - ▶ If there a path  $P$  is a solution to Jogger, then the subset of  $S$  with elements corresponding to the edges in  $P$  is a subset with sum  $i = t$  and thus is a solution to SS.  $\square$

