Logic Summary

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- **Giving credit where credit is due:**
  - **Most of slides for this lecture are based on slides created by Dr. Ben Choi, Louisiana Technical University.**
  - **I have modified them and added new slides**

Analysis Tool: Logic

- **Logic is a system for formalizing natural language statements so that we can reason more accurately.**
- **The simplest statements are called *atomic formulas*.**
- **More complex statements can be build up through the use of *logical connectives*: $\wedge$ "and", $\vee$ "or", $\neg$ "not", $\Rightarrow$ "implies" $A \Rightarrow B$ "A implies B" "if A then B"**
- **$A \Rightarrow B$ is logically equivalent to $\neg A \vee B$**
- **DeMorgan's laws:**
  - **$\neg (A \wedge B)$ is logically equivalent to $\neg A \vee \neg B$**
  - **$\neg (A \vee B)$ is logically equivalent to $\neg A \wedge \neg B$**

Quantifiers: all, some

- **“for all x” $\forall$x P(x) is true iff P(x) is true for *all* x**
  - **universal quantifier (universe of discourse)**
- **“there exist x” $\exists$x P(x) is true iff P(x) is true for *some* value of x**
  - **existential quantifier**
- **$\forall$x A(x) is logically equivalent to $\neg\exists$x($\neg$A(x))**
- **$\exists$x A(x) is logically equivalent to $\neg\forall$x($\neg$A(x))**
- **$\forall$x (A(x) $\Rightarrow$ B(x))**
  “For all x such that if A(x) holds then B(x) holds”

Prove by counterexample, Contraposition, Contradiction

- ***Counterexample***
  to prove $\forall x\ (A(x) \Rightarrow B(x))$ is false, we show *some* object x for which A(x) is true and B(x) is false.
  - $\neg(\forall x\ (A(x) \Rightarrow B(x))) \quad \Leftrightarrow \quad \exists x\ (A(x) \land \neg B(x))$
- ***Contraposition***
  to prove $A \Rightarrow B$, we show $(\neg B) \Rightarrow (\neg A)$
- ***Contradiction***
  to prove $A \Rightarrow B$, we assume $\neg B$ and then prove B.
  - $A \Rightarrow B \quad \Leftrightarrow \quad (A \land \neg B) \Rightarrow B$
  - $A \Rightarrow B \quad \Leftrightarrow \quad (A \land \neg B)$ is false
- **Assuming $(A \land \neg B)$ is true,**
  and discover a *contradiction* (such as $A \land \neg A$),
  then conclude $(A \land \neg B)$ is false, and so $A \Rightarrow B$.

Prove by Contradiction Example

- **Prove $[B \wedge (B \Rightarrow C)] \Rightarrow C$**
  - **by contradiction**
- **Proof:**

Assume $\neg C$

$$\neg C \wedge [B \wedge (B \Rightarrow C)]$$
$$\Rightarrow \neg C \wedge [B \wedge (\neg B \vee C)]$$
$$\Rightarrow \neg C \wedge [(B \wedge \neg B) \vee (B \wedge C)]$$
$$\Rightarrow \neg C \wedge [(B \wedge C)]$$
$$\Rightarrow \neg C \wedge C \wedge B$$

$\Rightarrow$ **False, *Contradiction***

$\Rightarrow$ **C**

Rules of Inference

- **A rule of inference is a *general pattern* that allows us to draw some new conclusion from a set of given statements.**
  - **If we know {…} then we can conclude {…}**
- **If {B and (B ⇒ C)} then {C}**
  - **modus ponens**
- **If {A ⇒ B and B ⇒ C} then {A ⇒ C}**
  - **syllogism**
- **If {B ⇒ C and ¬B ⇒ C} then {C}**
  - **rule of cases**

Boolean Algebra

A set B and two binary operators + "or, $\vee$", $\cdot$ "and, $\wedge$" creates a Boolean Algebra iff Huntington's Postulates hold:

1. **Closure:**
   - **if x, y $\in$ B and z = x + y then z $\in$ B**
   - **if x, y $\in$ B and z = x $\cdot$ y then z $\in$ B**
2. **Identity and Zero elements:**
   - **for + designated by 0: x + 0 = x**
   - **for $\cdot$ designated by 1: x $\cdot$ 1 = x**
3. **Commutative:**
   - **x + y = y + x**
   - **x $\cdot$ y = y $\cdot$ x**
4. **Distributive:**
   - **x $\cdot$ (y + z) = (x $\cdot$ y) + (x $\cdot$ z)**
   - **x + (y $\cdot$ z) = (x + y) $\cdot$ (x + z)**
5. **Complement: for every element x $\in$ B, there exits an element x' $\in$ B s.t.**
   - **x + x' = 1, x $\cdot$ x' = 0**
6. **There are at least two distinct elements in B**

Truth Table and Implication Tautology (theorem)

- **Show $[B \wedge (B \Rightarrow C)] \Rightarrow C$ is a tautology:**

B	C	$(B \Rightarrow C)$	$[B \wedge (B \Rightarrow C)]$	$[B \wedge (B \Rightarrow C)] \Rightarrow C$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

- **For every assignment for B and C,**
  - **the statement is True**

Prove by Rules of Inference

- **Prove $[B \wedge (B \Rightarrow C)] \Rightarrow C$**
  - **Proof:**

$$[B \wedge (B \Rightarrow C)] \Rightarrow C$$
$$\Rightarrow \neg[B \wedge (B \Rightarrow C)] \vee C$$
$$\Rightarrow \neg[B \wedge (\neg B \vee C)] \vee C$$
$$\Rightarrow \neg[(B \wedge \neg B) \vee (B \wedge C)] \vee C$$
$$\Rightarrow \neg [(B \wedge C)] \vee C$$
$$\Rightarrow \neg B \vee \neg C \vee C$$

$\Rightarrow$ **True (*tautology*)**

- **Direct Proof:**

$$[B \wedge (B \Rightarrow C)] \Rightarrow [B \wedge C] \Rightarrow C$$