JDEP 284H
Foundations of Computer Systems

Floating Point

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Giving credit where credit is due

- **Most of slides for this lecture are based on slides created by Drs. Bryant and O'Hallaron, Carnegie Mellon University.**
- **I have modified them and added new slides.**

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Topics

- **IEEE Floating Point Standard**
- **Rounding**
- **Floating Point Operations**
- **Mathematical properties**

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Floating Point Puzzles

- **For each of the following C expressions, either:**
  - **Argue that it is true for all argument values**
  - **Explain why not true**

```
int x = …;
float f = …;
double d = …;
```

Assume neither `d` nor `f` is NaN

- `x == (int)(float) x`
- `x == (int)(double) x`
- `f == (float)(double) f`
- `d == (float) d`
- `f == -(-f);`
- `2/3 == 2/3.0`
- `d < 0.0` ⇒ `((d*2) < 0.0)`
- `d > f` ⇒ `-f < -d`
- `d * d >= 0.0`
- `(d+f)-d == f`

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IEEE Floating Point

IEEE Standard 754

- **Established in 1985 as uniform standard for floating point arithmetic**
  - **Before that, many idiosyncratic formats**
- **Supported by all major CPUs**

Driven by Numerical Concerns

- **Nice standards for rounding, overflow, underflow**
- **Hard to make go fast**
  - **Numerical analysts predominated over hardware types in defining standard**

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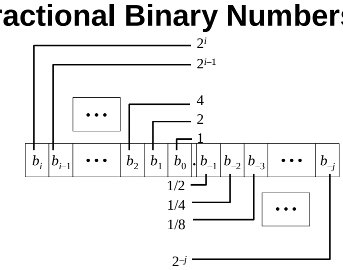
Fractional Binary Numbers



Representation

- **Bits to right of "binary point" represent fractional powers of 2**
- **Represents rational number:** $\sum_{k=-j}^{i} b_k \cdot 2^k$

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Frac. Binary Number Examples

Value	Representation
5 3/4	101.11_2
2 7/8	10.111_2
63/64	0.111111_2

Observations

- Divide by 2 by shifting right
- Multiply by 2 by shifting left
- Numbers of form $0.111111\ldots_2$ just below 1.0
  - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
  - Use notation $1.0 - \varepsilon$

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Representable Numbers

Limitation

- Can only exactly represent numbers of the form $x/2^k$
- Other numbers have repeating bit representations

Value	Representation
1/3	$0.0101010101[01]\ldots_2$
1/5	$0.001100110011[0011]\ldots_2$
1/10	$0.0001100110011[0011]\ldots_2$

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Floating Point Representation

Numerical Form

- $-1^s\ M\ 2^E$
  - Sign bit *s* determines whether number is negative or positive
  - Significand *M* normally a fractional value in range [1.0,2.0).
  - Exponent *E* weights value by power of two

Encoding

s	exp	frac

- MSB is sign bit
- `exp` field encodes *E*
- `frac` field encodes *M*

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Floating Point Precisions

Encoding

s	exp	frac

- MSB is sign bit
- `exp` field encodes *E*
- `frac` field encodes *M*

Sizes

- Single precision: 8 `exp` bits, 23 `frac` bits
  - 32 bits total
- Double precision: 11 `exp` bits, 52 `frac` bits
  - 64 bits total
- Extended precision: 15 `exp` bits, 63 `frac` bits
  - Only found in Intel-compatible machines
  - Stored in 80 bits
    - » 1 bit wasted

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"Normalized" Numeric Values

Condition

- `exp` ≠ `000…0` and `exp` ≠ `111…1`

Exponent coded as *biased* value

E = *Exp* – *Bias*

- *Exp* : unsigned value denoted by `exp`
- *Bias* : Bias value
  - » Single precision: 127 (*Exp*: 1…254, *E*: -126…127)
  - » Double precision: 1023 (*Exp*: 1…2046, *E*: -1022…1023)
  - » in general: *Bias* = 2^{e-1} - 1, where e is number of exponent bits

Significand coded with implied leading 1

M = $1.xxx\ldots x_2$

- `xxx…x`: bits of `frac`
- Minimum when `000…0` (*M* = 1.0)
- Maximum when `111…1` (*M* = 2.0 – ε)
- Get extra leading bit for "free"

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Normalized Encoding Example

Value

`Float F = 15213.0;`

- $15213_{10} = 11101101101101_2 = 1.1101101101101_2 \times 2^{13}$

Significand

M = $1.\underline{1101101101101}_2$

`frac` = $\underline{1101101101101}0000000000_2$

Exponent

E = 13

Bias = 127

Exp = 140 = 10001100_2

Floating Point Representation (from Lecture 2):

Hex:	4	6	6	D	B	4	0	0
Binary:	0100	0110	0110	1101	1011	0100	0000	0000
140:	100	0110	0					
15213:			1110	1101	1011	01		

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Denormalized Values

Condition
- `exp` = `000…0`

Value
- Exponent value *E* = –*Bias* + 1
- Significand value *M* = `0.xxx…x`$_2$
  - `xxx…x`: bits of `frac`

Cases
- `exp` = `000…0`, `frac` = `000…0`
  - Represents value 0
  - Note that have distinct values +0 and –0
- `exp` = `000…0`, `frac` ≠ `000…0`
  - Numbers very close to 0.0
  - Lose precision as get smaller
  - "Gradual underflow"

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Special Values

Condition
- `exp` = `111…1`

Cases
- `exp` = `111…1`, `frac` = `000…0`
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., 1.0/0.0 = –1.0/–0.0 = +∞, 1.0/–0.0 = –∞
- `exp` = `111…1`, `frac` ≠ `000…0`
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ – ∞

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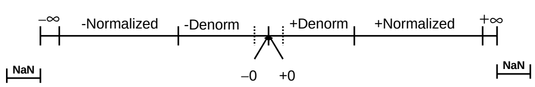
Summary of Floating Point Real Number Encodings



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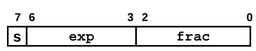
Tiny Floating Point Example

8-bit Floating Point Representation
- the sign bit is in the most significant bit.
- the next four bits are the exponent, with a bias of 7.
- the last three bits are the `frac`

- **Same General Form as IEEE Format**
  - normalized, denormalized
  - representation of 0, NaN, infinity



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Values Related to the Exponent

Exp	exp	E	2^E	
0	0000	-6	1/64	(denorms)
1	0001	-6	1/64	
2	0010	-5	1/32	
3	0011	-4	1/16	
4	0100	-3	1/8	
5	0101	-2	1/4	
6	0110	-1	1/2	
7	0111	0	1	
8	1000	+1	2	
9	1001	+2	4	
10	1010	+3	8	
11	1011	+4	16	
12	1100	+5	32	
13	1101	+6	64	
14	1110	+7	128	
15	1111	n/a		(inf, NaN)

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Dynamic Range

	s	exp	frac	*E*	Value	
Denormalized numbers	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	← closest to zero
	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	← largest denorm
Normalized numbers	0	0001	000	-6	8/8*1/64 = 8/512	← smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	← closest to 1 below
	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	← closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	← largest norm
	0	1111	000	n/a	inf	

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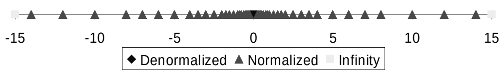
Distribution of Values

6-bit IEEE-like format

- e = 3 exponent bits
- f = 2 fraction bits
- Bias is 3

Notice how the distribution gets denser toward zero.



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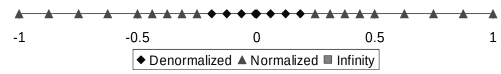
Distribution of Values (close-up view)

6-bit IEEE-like format

- e = 3 exponent bits
- f = 2 fraction bits
- Bias is 3



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Interesting Numbers

Description	`exp`	`frac`	**Numeric Value**
Zero	00…00	00…00	0.0
Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
■ Single ≈ 1.4×10^{-45}			
■ Double ≈ 4.9×10^{-324}			
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
■ Single ≈ 1.18×10^{-38}			
■ Double ≈ 2.2×10^{-308}			
Smallest Pos. Normalized	00…01	00…00	$1.0 \times 2^{-\{126,1022\}}$
■ Just larger than largest denormalized			
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
■ Single ≈ 3.4×10^{38}			
■ Double ≈ 1.8×10^{308}			

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Special Properties of Encoding

FP Zero Same as Integer Zero

- All bits = 0

Can (Almost) Use Unsigned Integer Comparison

- Must first compare sign bits
- Must consider -0 = 0
- NaNs problematic
  - Will be greater than any other values
  - What should comparison yield?
- Otherwise OK
  - Denorm vs. normalized
  - Normalized vs. infinity

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Floating Point Operations

Conceptual View

- First compute exact result
- Make it fit into desired precision
  - Possibly overflow if exponent too large
  - Possibly round to fit into `frac`

Rounding Modes (illustrate with $ rounding)

	\$1.40	**\$1.60**	**\$1.50**	**\$2.50**	**–\$1.50**
■ **Zero**	\$1	\$1	\$1	\$2	–\$1
■ **Round down (-∞)**	\$1	\$1	\$1	\$2	–\$2
■ **Round up (+∞)**	\$2	\$2	\$2	\$3	–\$1
■ **Nearest Even** (default)	\$1	\$2	\$2	\$2	–\$2

Note:
1. Round down: rounded result is close to but no greater than true result.
2. Round up: rounded result is close to but no less than true result.

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Closer Look at Round-To-Even

Default Rounding Mode

- Hard to get any other kind without dropping into assembly
- All others are statistically biased
  - Sum of set of positive numbers will consistently be over- or under-estimated

Applying to Other Decimal Places / Bit Positions

- When exactly halfway between two possible values
  - Round so that least significant digit is even
- E.g., round to nearest hundredth

1.2349999	1.23	(Less than half way)
1.2350001	1.24	(Greater than half way)
1.2350000	1.24	(Half way—round up)
1.2450000	1.24	(Half way—round down)

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Rounding Binary Numbers

Binary Fractional Numbers

- **"Even" when least significant bit is `0`**
- **Half way when bits to right of rounding position = `100…`$_2$**

Examples

- **Round to nearest 1/4 (2 bits right of binary point)**

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

Operands

$(-1)^{s1}\ M1\ \ 2^{E1}$ * $(-1)^{s2}\ M2\ \ 2^{E2}$

Exact Result

$(-1)^{s}\ M\ \ 2^{E}$

- **Sign** *s*: *s1 ^ s2*
- **Significand** *M*: *M1 * M2*
- **Exponent** *E*: *E1 + E2*

Fixing

- **If** *M* 2, **shift** *M* **right, increment** *E*
- **If** *E* **out of range, overflow**
- **Round** *M* **to fit `frac` precision**

Implementation

- **Biggest chore is multiplying significands**

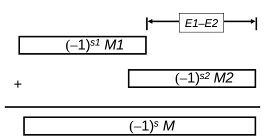
FP Addition

Operands

$(-1)^{s1}\ M1\ \ 2^{E1}$

$(-1)^{s2}\ M2\ \ 2^{E2}$

- **Assume** *E1* **>** *E2*



Exact Result

$(-1)^{s}\ M\ \ 2^{E}$

- **Sign** *s*, **significand** *M*:
  - **Result of signed align & add**
- **Exponent** *E*: *E1*

Fixing

- **If** *M* 2, **shift** *M* **right, increment** *E*
- **if** *M* < 1, **shift** *M* **left** *k* **positions, decrement** *E* **by** *k*
- **Overflow if** *E* **out of range**
- **Round** *M* **to fit `frac` precision**

Mathematical Properties of FP Add

Compare to those of Abelian Group

- **Closed under addition?** **YES**
  - **But may generate infinity or NaN**
- **Commutative?** **YES**
- **Associative?** **NO**
  - **Overflow and inexactness of rounding**
- **0 is additive identity?** **YES**
- **Every element has additive inverse** **ALMOST**
  - **Except for infinities & NaNs**

Monotonicity

- *a* *b* ⇒ *a+c* *b+c*? **ALMOST**
  - **Except for infinities & NaNs**

Math. Properties of FP Mult

Compare to Commutative Ring

- **Closed under multiplication?** **YES**
  - **But may generate infinity or NaN**
- **Multiplication Commutative?** **YES**
- **Multiplication is Associative?** **NO**
  - **Possibility of overflow, inexactness of rounding**
- **1 is multiplicative identity?** **YES**
- **Multiplication distributes over addition?** **NO**
  - **Possibility of overflow, inexactness of rounding**

Monotonicity

- *a* *b* & *c* 0 ⇒ *a* * *c* *b* * *c*? **ALMOST**
  - **Except for infinities & NaNs**

Floating Point in C

C Guarantees Two Levels

`float` **single precision**

`double` **double precision**

Conversions

- **Casting between `int`, `float`, and `double` changes numeric values**
- `Double` **or** `float` **to** `int`
  - **Truncates fractional part**
  - **Like rounding toward zero**
  - **Not defined when out of range**
    - » **Generally saturates to TMin or TMax**
- `int` **to** `double`
  - **Exact conversion, as long as int has 53 bit word size**
- `int` **to** `float`
  - **Will round according to rounding mode**

Answers to Floating Point Puzzles

```
int x = …;
float f = …;
double d = …;
```

Assume neither `d` nor `f` is NAN

- `x == (int)(float) x` — **No: 24 bit significand**
- `x == (int)(double) x` — **Yes: 53 bit significand**
- `f == (float)(double) f` — **Yes: increases precision**
- `d == (float) d` — **No: loses precision**
- `f == -(-f);` — **Yes: Just change sign bit**
- `2/3 == 2/3.0` — **No: 2/3 == 0**
- `d < 0.0 ⇒ ((d*2) < 0.0)` — **Yes!**
- `d > f ⇒ -f < -d` — **Yes!**
- `d * d >= 0.0` — **Yes!**
- `(d+f)-d == f` — **No: Not associative**

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Ariane 5

- **Exploded 37 seconds after liftoff**
- **Cargo worth $500 million**

Why

- **Computed horizontal velocity as floating point number**
- **Converted to 16-bit integer**
- **Worked OK for Ariane 4**
- **Overflowed for Ariane 5**
  - **Used same software**

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Summary

IEEE Floating Point Has Clear Mathematical Properties

- **Represents numbers of form $M \times 2^E$**
- **Can reason about operations independent of implementation**
  - **As if computed with perfect precision and then rounded**
- **Not the same as real arithmetic**
  - **Violates associativity/distributivity**
  - **Makes life difficult for compilers & serious numerical applications programmers**

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